A GEOMETRICAL BACKGROUND FOR DE SITTER'S WORLD

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"He's in prison now, being punished: and the trial doesn't even begin till next Wednesday: and of course the crime comes last of all." The White Queen [1].

1. Introduction. Through a natural misunderstanding, students who have just begun to learn about non-euclidean geometry are apt to say that the elliptic plane can be developed on a sphere, and the hyperbolic plane on a hyperboloid. "Of one sheet, or two sheets?" you ask, and then the statement is admitted to have been confused. But let us see whether there is after all a germ of truth in it.

We must remember that the classification of quadrics as ellipsoids, paraboloids (of two kinds), and hyperboloids (of two kinds) is not peculiar to euclidean geometry, but belongs properly to the wider field of *affine* geometry. All the theorems of affine geometry hold not only in euclidean geometry but also in Minkowskian (or pseudo-euclidean) geometry, where every point is the vertex of a real null cone (or isotropic cone, or cone of minimal lines). A hyperboloid whose asymptotic cone is of this kind has the property that all its points are equidistant from its center; it may thus legitimately be called a sphere. In other words, a euclidean sphere is an ellipsoid, but a Minkowskian sphere is a hyperboloid—of one or two sheets according as the radius is space-like or time-like. The intrinsic geometry of either sheet of the sphere of two sheets is indeed hyperbolic. The sphere of one sheet is quite different, but equally worthy of study; that is where de Sitter's World comes in.

These ideas are not so familiar as they deserve to be, though of course they are well known to experts. It is hoped that the present note will help to clarify the situation by emphasizing the precise relationship of the various geometries. We shall be chiefly concerned with spaces of two or three dimensions; but the extension to more than three dimensions presents no difficulty, and will be seen to have an interesting application in relativistic cosmology.

2. Affine geometry. Affine geometry deals with points, lines, and planes, having the familiar properties of serial order, continuity, and parallelism. (See [2], pp. 161, 162, 186; or [8], pp. 51, 52, 56, 57.) With the aid of parallelograms we can construct the midpoint of any given segment, and describe a *translation*. Since there is a unique line through a given point parallel to a given line, and a unique plane through a given point parallel to a given plane, we can say that every line belongs to a definite *bundle of parallels*, and every plane to a definite *pencil of parallels*. Such bundles and pencils are said to be improper, in contrast to the proper bundle of lines through a point, and the proper pencil of planes through a line. By treating bundles and pencils as points and lines, with the natural interpretation for the relation of incidence, we obtain *real projective geometry*. ([2], pp. 165–174.) Since the *proper* bundles and pencils correspond in an obvious manner to the ordinary points and lines, we can think of the projective space as an extension of the affine space. The extra entities, namely the

bundles and pencils of parallels, are then called *points at infinity* and *lines at infinity*. In the projective geometry these are the points and lines of a plane: *the plane at infinity*.

Conversely, we could begin with real projective geometry, in which all planes are alike, and agree to single out one of them as the plane at infinity, letting any two lines or planes be considered parallel if they intersect on that plane. In this sense, affine space is derived from real projective space by *removing* one plane (with all its points and lines).

Quadrics can be defined projectively, as by von Staudt and Enriques ([2], p. 68) or Seydewitz and Reye ([2], p. 219). We can then distinguish the *ruled* quadrics from the non-ruled or *oval* quadrics. In affine geometry, these can be further classified according to the nature of their intersection with the plane at infinity:

Oval Quadric	*Ruled Quadric*	*Section at Infinity*
Ellipsoid	—	Nothing
Elliptic paraboloid	—	One point
—	Hyperbolic paraboloid	Two lines
Hyperboloid of two sheets	Hyperboloid of one sheet	Conic

The *center* of a quadric is the pole of the plane at infinity. Thus the center of a paraboloid is its point of contact with the plane at infinity. In the other cases the center is an ordinary point, and we can alternatively classify the quadrics according to the manner in which they meet their diameters and diametral planes. Every diameter of an ellipsoid meets the surface twice. But a hyperboloid has some diameters that do not meet it at all. In the case of a hyperboloid of one sheet, any diametral plane meets every generator; so the section of the surface by such a plane is a conic. On the other hand, a hyperboloid of two sheets has some diametral planes that do not meet it at all; the two sheets lie on opposite sides of any such plane.

As an alternative to this synthetic treatment, we can build up a theory of vectors—added, subtracted, or multiplied by scalars—and use any three independent vectors to describe a system of affine *coordinates*. (These are illustrated by oblique Cartesian coordinates with different units of measurement along the three axes.) Then every equation of the first degree represents a plane, and certain equations of the second degree represent quadrics. The extension to projective space is achieved by the familiar device of rendering the coordinates homogeneous, *i.e.*, replacing x_0, x_1, x_2 by $x_0/x_3, x_1/x_3, x_2/x_3$. Then the plane at infinity is $x_3 = 0$.

3. The absolute polarity. We have seen that the plane at infinity of affine space is a projective plane. We know that the geometry of a bundle in euclidean space (with its lines and planes interpreted as points and lines) is elliptic, which means that the plane at infinity of euclidean space is an elliptic plane. In the Appendix to [6], Robb shows that the geometry of a bundle in Minkowskian space (or space-time) is hyperbolic, which means that the plane at infinity of

Minkowskian space (of 2+1 dimensions) is a hyperbolic plane. It is perhaps worth while to outline the converse process, and show how an elliptic or hyperbolic metric in the plane at infinity of affine space induces a euclidean or Minkowskian metric in the whole space. For this purpose, we single out an *absolute polarity* (elliptic or hyperbolic) in the plane at infinity, and consider as *perpendicular* any line and plane whose elements at infinity correspond in this polarity.

Given two points B and C, the *sphere* on BC as diameter may be defined as the locus of the point of intersection of a variable line through B with the perpendicular plane through C (or vice versa). Thus a sphere is a quadric, according to Seydewitz's construction. To see what kind of quadric this is, we investigate the possibility of its containing a point at infinity, say P. In such a case the line BP would be perpendicular to a certain plane through CP, and P would lie on its own polar. This can happen only if the polarity is hyperbolic; the locus of such *self-conjugate* points is then a conic, as defined by von Staudt. Thus the sphere is an ellipsoid or a hyperboloid according as the absolute polarity is elliptic or hyperbolic. (A nice justification for the accepted terminology!)

The properties of congruence may be developed by defining the reflection in a plane α as the harmonic homology with axial plane α and center the point at infinity on lines perpendicular to α. A displacement is then defined as the product of any even number of reflections; in particular, the product of reflections in two parallel planes is seen to be a translation. Given two points O and C, the sphere through C with center O is the locus of the image of C by reflection in all the planes through O. (This clearly agrees with the above construction, if O is the mid point of BC.) It follows that all the radii of a sphere are congruent.

If we now refer to a standard list of axioms of congruence for euclidean geometry (*e.g.* [2], p. 180), we find that they are all valid, *provided the absolute polarity is elliptic*. Consider, for instance, the first axiom:

> If A and B are distinct points, then on any ray C/E there is just one point D such that $AB \equiv CD$.

To locate the desired point D, we apply the translation that takes A to C; if this takes B to B', we draw the sphere through B' with center C, to meet the given ray at D. This construction must succeed in the elliptic case, as the sphere then has radii in all directions. But in the hyperbolic case it will fail whenever CE happens to be one of the exterior diameters.

Proceeding similarly with the remaining axioms, we find that the elliptic polarity does indeed give rise to euclidean geometry. (See Veblen [10], pp. 287-302.) In order to show that the hyperbolic polarity gives rise to Minkowskian geometry, we could similarly test the postulates of Robb [6].* But the details of the work are rather formidable, so we shall be content to interpret the various entities and appeal to coordinates for the actual identification.

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* Since we are at present limiting the number of dimensions to three (*i.e.* 2+1, instead of 3+1), the relevant Postulates are I–XVIII, XXI, and the denial of XIX (cf. XX).

4. Minkowskian geometry. In the plane at infinity with a hyperbolic polarity we have a conic, the *Absolute*, which is the locus of self-conjugate points and the envelope of self-conjugate lines. The remaining points and lines at infinity are thereby separated into two classes: interior and exterior points, secants and exterior lines. ([2], p. 56.) In order to examine the induced metric, we classify the lines and planes of the affine space according to their sections by the plane at infinity. (We use the modern names for them, with Robb's in parentheses.)

Line or Plane	*Section at Infinity*
Time-like line (inertia line)	Interior point
Null line (optical line)	Point on the Absolute
Space-like line (separation line)	Exterior point
Minkowskian plane (inertia plane)	Secant
Null plane (optical plane)	Tangent
Euclidean plane (separation plane)	Exterior line

We have seen that in this geometry a sphere is a hyperboloid. Strictly, we should include the case when the hyperboloid degenerates into a cone. In fact, if OC (or BC) is a null line, the locus defined in §3 is the *null cone* which joins O to the Absolute. In the remaining (non degenerate) cases, the radius OC may be either space-like or time-like. The other radii will follow suit, as a line of either kind reflects into a line of the same kind. We proceed to show that a non degenerate sphere is a hyperboloid of *one* or *two* sheets according as its radii are space-like or time-like.

If the radii are space-like, their points at infinity are exterior. Since every line at infinity contains some exterior points, the sphere meets each of its diametral planes. (The curve of intersection is an ordinary circle if the plane is euclidean, two parallel lines if it is a null plane, and in the remaining case a Minkowskian "circle," which is a special hyperbola of the affine geometry.)

If, on the other hand, the radii are time-like, their points at infinity are interior, whereas every point on an exterior line is exterior. Thus a euclidean plane through the center will not meet such a sphere, but will separate the two sheets.

By arbitrarily labelling the two sheets as first and second, or "past" and "future," we assign a definite sense along the time-like diameters. By applying translations, we deduce a consistent sense along *all* time-like lines. Then, for any two points whose join is time-like, we can assert that one is *after* the other.

Two null lines through a point O determine a Minkowskian plane which they decompose into four angular regions, one of which consists of points *after* O. The rays which bound this region determine a definite sense along the two null lines, and so along all null lines. Thus the idea of one point being after another applies to points on a null line as well as to points on a time-like line.

Robb's "conical order" is now established: there is at every point a null cone, whose generators are parallel to respective generators of any other null cone, and a time-like line is interior to the null cone at each of its points. This is essentially the model considered by Robb ([6], pp. 16–22); but by defining it in affine space instead of euclidean, we avoid the "distortion" which he mentions on p. 402.

For the analytical treatment, we choose projective coordinates in the plane at infinity in such a manner that the hyperbolic polarity takes the canonical form

$$X_0 = -x_0, \qquad X_1 = x_1, \qquad X_2 = x_2. \tag{4.1}$$

Then the Absolute has the ordinary equation

$$-x_0^2 + x_1^2 + x_2^2 = 0$$

and the tangential equation $-X_0^2+X_1^2+X_2^2=0$. ([2], pp. 86, 87.) The triangle of reference is a self-polar triangle, and may be joined to an arbitrary origin to form a suitable trihedron of reference for affine coordinates. We can use the same symbols for these coordinates by letting $(l_0, l_1, l_2, 0)$ denote the point previously called (l_0, l_1, l_2), which is the point at infinity on all lines parallel to

$$\frac{x_0}{l_0} = \frac{x_1}{l_1} = \frac{x_2}{l_2}. \tag{4.2}$$

These lines are time-like, null, or space-like, according as the number $l_0^2-l_1^2-l_2^2$ is positive, zero, or negative; and similarly the plane

$$X_0x_0 + X_1x_1 + X_2x_2 = 0$$

is euclidean, null, or Minkowskian, according to the sign of $X_0^2-X_1^2-X_2^2$. Applying the polarity (4.1), we see that the line (4.2) is perpendicular to the plane

$$-l_0x_0 + l_1x_1 + l_2x_2 = 0$$

(and to any parallel plane, which can be derived by replacing the zero on the right by another constant).

Let the points B and C of §3 have the affine coordinates $(-y_0, -y_1, -y_2)$ and (y_0, y_1, y_2), and let (x_0, x_1, x_2) be the point where the line

$$\frac{x_0 + y_0}{l_0} = \frac{x_1 + y_1}{l_1} = \frac{x_2 + y_2}{l_2}$$

through B meets the perpendicular plane

$$-l_0(x_0 - y_0) + l_1(x_1 - y_1) + l_2(x_2 - y_2) = 0$$

through C. Then the sphere described by the point (x) has the equation

$$-x_0^2 + x_1^2 + x_2^2 = -y_0^2 + y_1^2 + y_2^2,$$

and so is a hyperboloid of one sheet, a null cone, or a hyperboloid of two sheets, according to the sign of the constant on the right.

In pure affine geometry we can compare distances along any one line, or along parallel lines, but the comparison of distances along non-parallel lines simply has no meaning. However, now that we have metricized the space by defining spheres, we can compare distances along any two time-like lines, or along any two space-like lines; but it is still impossible to compare distances along lines of different kinds, or along null lines with different directions, and

we shall not attempt to do so. Accordingly, we define distances (or *intervals*) of two kinds, so as to be able to say that the surface

$$-x_0^2+x_1^2+x_2^2=s^2$$

is a sphere of space-like radius s, while

$$x_0^2-x_1^2-x_2^2=t^2$$

is a sphere of time-like radius t.

It is sometimes convenient to unify the formulae by saying that any point (x_0, x_1, x_2) is distant $\sqrt{x_0^2-x_1^2-x_2^2}$ from the origin, in which case a time-like interval appears as a real number, and a space-like interval as a pure-imaginary number. The opposite convention could be used just as well, and the old-fashioned word "imaginary" should not give the subject any air of mystery. The modern view of a complex number, as an ordered pair of real numbers, clarifies the situation and explains the mutual incomparability of the two kinds of interval: no real multiple of $[t, 0]$ can equal $[0, s]$ with $s\neq 0$.

We observe also that in this scheme every distance along a null line is zero. (That is why it is called a null line.) Thus a null cone may be considered as a sphere of zero radius.

As the metric which we have obtained agrees with that defined by Minkowski [5], the desired identification is now complete. On the one hand, the geometry of Minkowski's time-like lines, or of their points at infinity, is hyperbolic; on the other hand, affine space acquires a Minkowskian metric as soon as we specialize a hyperbolic polarity at infinity.

The fact that we have limited the number of dimensions to $1+2$ is unimportant, as the extension to $1+3$ dimensions can be made without difficulty; *e.g.* the interval from $(0, 0, 0, 0)$ to (x_0, x_1, x_2, x_3) is $\sqrt{x_0^2-x_1^2-x_2^2-x_3^2}$. (This x_3 is on a par with the x_1 and x_2, and has nothing to do with the x_3 of §2.) The $(1+2)$-dimensional space is simply a section of the $(1+3)$-dimensional space. Further extensions can be made just as easily.

5. The representation of non-euclidean geometries on spheres. In three-dimensional euclidean space, where the plane at infinity is elliptic, there is a $(1, 1)$ correspondence between points at infinity and diameters of a sphere, and consequently a $(1, 2)$ correspondence between points at infinity and points on a sphere. The elliptic plane cannot be mapped on a sphere without duplication. If we try to map it on a hemisphere, we still have trouble on the peripheral circle. In other words, points at infinity correspond to diameters, whereas points on the sphere correspond to radii; and there is no satisfactory way of associating all the diameters with half the radii.

In $(1+2)$-dimensional Minkowskian space, where the plane at infinity is hyperbolic, there is similarly a $(1, 2)$ correspondence between points at infinity (interior and exterior) and points on a sphere (of time-like or space-like radius, respectively). In the case of exterior points the situation is just as before: the

sphere has a single sheet, and does not provide a (1, 1) mapping of the exterior points at infinity unless we agree to identify each pair of antipodal points of the sphere.

But the case of interior points is quite different. There is still a (1, 2) correspondence between those points at infinity and the points on a two-sheeted sphere; but now any pair of antipodal points consists of one point on each sheet, so there is a (1, 1) correspondence between interior points at infinity and points on either sheet. These interior points are just the "ordinary" points of the hyperbolic plane. Hence *either sheet of a sphere of two sheets provides an undistorted map of the proper hyperbolic plane*, without any duplication. Each line of the hyperbolic plane is represented by a diametral section of the sheet, namely by one branch of a hyperbola. This hyperbola is a Minkowskian circle—a "great circle" of the sphere. Such a curve on the surface is a geodesic in the ordinary sense of "shortest path," as the tangent planes to this kind of sphere are euclidean.

These results are obvious analytically, when we describe hyperbolic geometry in terms of coordinates x_0, x_1, x_2 satisfying

$$x_0^2 - x_1^2 - x_2^2 = 1, \qquad x_0 > 0,$$

with the rule that the distance between points (x) and (y) is

$$\text{arg cosh } (x_0y_0 - x_1y_1 - x_2y_2),$$

so that the line element is given by

$$ds^2 = -dx_0^2 + dx_1^2 + dx_2^2.$$

([2], pp. 209, 248.)

Eddington ([4], p. 165) has shown that the sphere of *one* sheet in (1+2)-dimensional Minkowskian space provides a model for the (1+1)-dimensional de Sitter's world,* wherein coordinates x_0, x_1, x_2 can be found such that

$$-x_0^2 + x_1^2 + x_2^2 = R^2,$$

with the line element given by

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2.$$

His lucid discussion of elliptic space ([4], pp. 157–158) is relevant here, and indicates that, in order to obtain an unduplicated model of the de Sitter plane, we should identify antipodal points of the one-sheeted sphere.

But we have already obtained the same model for the "exterior-hyperbolic" plane, whose points are the absolute poles of the lines of the ordinary hyperbolic plane. Adding two dimensions, *de Sitter's world is four-dimensional exterior-hyperbolic space*—the space whose points are exterior to an oval quadric three-

* Eddington clearly implies that the analogous manifold in (1+4)-dimensional Minkowskian space represents the (1+3)-dimensional de Sitter's world. Robertson ([7], p. 840) obtains the same result in a different way; but instead of "Minkowskian" he calls the underlying space "euclidean."

fold in projective four-space, with the metric determined by that quadric threefold as Absolute. (This remark was made to me about 1930 by Professor Patrick Du Val, but I have not seen it in print.) The three kinds of line—time-like, null, and space-like—are recognizable in hyperbolic geometry as the ordinary lines, lines at infinity, and ultra-infinite lines, *i.e.* as the secants, tangents, and exterior lines to the Absolute.

6. Projective coordinates. Returning, for simplicity, to the two-dimensional world, we now have *homogeneous* coordinates x_0, x_1, x_2 with

$$-x_0^2+x_1^2+x_2^2>0.$$

Retaining the convention whereby time-like and space-like intervals are regarded as real and pure-imaginary, respectively, we find (as in [2], p. 209) that the interval from (x) to (y) is

$$R \operatorname{arg\,cosh} \frac{|-x_0y_0+x_1y_1+x_2y_2|}{\sqrt{-x_0^2+x_1^2+x_2^2}\sqrt{-y_0^2+y_1^2+y_2^2}}. \tag{6.1}$$

This is time-like or space-like according as

$$-(x_1y_2-x_2y_1)^2+(x_2y_0-x_0y_2)^2+(x_0y_1-x_1y_0)^2$$

is positive or negative. In the former case the interval increases to infinity as the point (x) approaches the Absolute,

$$-x_0^2+x_1^2+x_2^2=0.$$

In the latter case the interval is preferably expressed as

$$iR \arccos \frac{|-x_0y_0+x_1y_1+x_2y_2|}{\sqrt{-x_0^2+x_1^2+x_2^2}\sqrt{-y_0^2+y_1^2+y_2^2}}.$$

Thus, as Dirac remarks ([3], p. 658), all space-like lines are finite.

To describe the manner in which Minkowski's metric arises as a limiting case of de Sitter's, we make use of the homogeneity of our coordinates by arbitrarily putting $x_1=R$. We are then left with non-homogeneous coordinates x_0, x_2 (or x_0, x_2, x_3, x_4), such that the interval from (x_0, x_2) to (y_0, y_2) is

$$\begin{aligned} &R \operatorname{arg\,cosh} \frac{R^2-x_0y_0+x_2y_2}{\sqrt{R^2-x_0^2+x_2^2}\sqrt{R^2-y_0^2+y_2^2}} \\ &= R \operatorname{arg\,sinh} \sqrt{\frac{R^2(x_0-y_0)^2-R^2(x_2-y_2)^2+(x_2y_0-x_0y_2)^2}{(R^2-x_0^2+x_2^2)(R^2-y_0^2+y_2^2)}}. \end{aligned} \tag{6.2}$$

It follows (as in [2], p. 255) that the line element is given by

$$ds^2=\frac{dx_0^2-dx_2^2+(x_2dx_0-x_0dx_2)^2/R^2}{\{1-(x_0^2-x_2^2)/R^2\}^2}.$$

When R tends to infinity, the interval becomes

$$\sqrt{(x_0 - y_0)^2 - (x_2 - y_2)^2},$$

and the line element takes its usual form for the Minkowskian plane, with time x_0 and distance x_2.

This result suggests the possibility of giving x_0 and x_2 the same interpretation *before* making R tend to infinity. What the cosmologists actually do, however, is to define an "angular" distance χ and time t, which are connected with our x_0, x_1, x_2 by the relations

$$x_0 = R \cos \chi \sinh t, \qquad x_1 = R \cos \chi \cosh t, \qquad x_2 = R \sin \chi$$

(or $x_0 : x_1 : x_2 : : \sinh t : \cosh t : \tan \chi$). Then the interval from (χ, t) to (χ', t') is

$$R \arg \cosh \{\cos \chi \cos \chi' \cosh (t - t') + \sin \chi \sin \chi'\}, \tag{6.3}$$

which reduces to $R|t - t'|$ when $\chi = \chi' = 0$, and to $iR|\chi - \chi'|$ when $t = t'$.

The world line of an undisturbed particle, initially at the origin ($x_0 = x_2 = 0$) is a straight line of the form $x_2 = ux_0$, or

$$\tan \chi = u \sinh t.$$

Thus the particle recedes from the origin as t increases. This is the phenomenon which gives de Sitter's world its chief experimental support, in view of the recession of nebulae. ([4], p. 162.)

7. Triangles in de Sitter's world. Exterior-hyperbolic geometry has been described by E. Study [9]. In particular, he discusses the possibility of one side of a triangle being greater than the sum of the other two. For such a comparison to have any meaning, the three sides must be all time-like or all space-like. If, further, we restrict consideration to triangles whose interior points are all exterior to the Absolute, we find that Study's results are expressible in the following form: there is *always* one side greater than the sum of the other two, say

$$AC > AB + BC.$$

This may be proved very simply in terms of the coordinates χ and t, by taking the line AC and the perpendicular line through B to be $\chi = 0$ and $t = 0$, or *vice versa*. The three sides, when time-like, determine three chords of the Absolute; the chord determined by AC is, in an obvious sense, *between* the other two. (See Fig. 1.) In the other case the Absolute lies entirely within one of the three "colunar" triangles, namely the one that shares the side AC of the triangle under consideration. (See Fig. 2.)

Thus a triangle with time-like sides can be expressed in the form

$$A(0, t), \qquad B(\chi, 0), \qquad C(0, -t'),$$

where χ, t, t' are positive. Then the sides are

$$AB = \text{arg cosh } (\cos \chi \cosh t) < t,$$
$$BC = \text{arg cosh } (\cos \chi \cosh t') < t',$$

and

$$AC = t + t'.$$

Eddington has given the following practical illustration of this result: Twin brothers, one of whom stays at home while the other travels very fast in various directions, will find when they meet again that the sedentary one has aged more than the traveller.

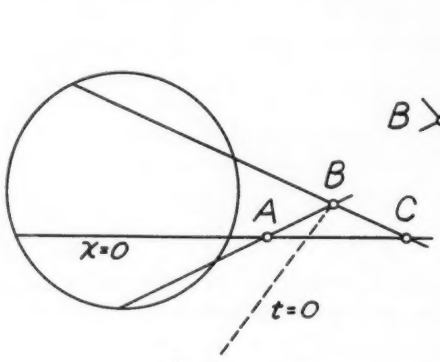

FIG. 1

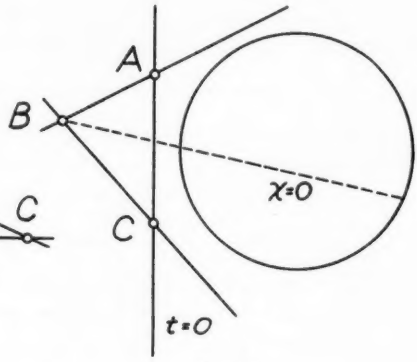

FIG. 2

Similarly, a triangle with space-like sides can be expressed as

$$A(\chi, 0), \qquad B(0, t), \qquad C(-\chi', 0),$$

with χ, χ', t positive. Dropping the i from the expression

$$iR \text{ arc cos } \{\cos \chi \cos \chi' \cosh (t - t') + \sin \chi \sin \chi'\}$$

for a space-like interval, we now have

$$AB = \text{arc cos } (\cos \chi \cosh t) < \chi,$$
$$BC = \text{arc cos } (\cos \chi' \cosh t) < \chi',$$

and

$$AC = \chi + \chi'.$$

Thus the straight path from A to C is longer than any other (everywhere space-like) path from A to C. In fact, the crooked distance can be made as small as we please by taking the origin to be the mid-point of AC (so that $\chi' = \chi$), and choosing B so that $\cosh t$ is nearly as great as $\sec \chi$.

When we add the extra dimensions for de Sitter's four-dimensional world, we find that a space-like line is neither minimum nor maximum. ([9], p. 115;

cf. [6], p. 3.) In fact, such a line, besides lying in de Sitter planes (*i.e.* secant planes to the Absolute) lies also in elliptic planes (exterior to the Absolute).

8. Through the looking-glass. In describing the two-dimensional de Sitter's world, Eddington writes of the totality of an observer's experience as forming a "lune." ([4], p. 165. His "90°" seems to have crept in through a false analogy with euclidean space.) We now recognize this lune as the angular region between two null lines (tangents to the Absolute). The corresponding figure in the whole four-dimensional world is the interior of the observer's null cone (which is an enveloping cone to the Absolute). The "horizon" which worried de Sitter is thus seen to be the section of the Absolute by the polar hyperplane of the observer. This evidently changes with the observer, like a rainbow. Its illusory character is further revealed by the fact that the world is strictly *exterior* to the Absolute, so that the horizon is not really a part of the world at all.

On the other hand, a rather disturbing paradox remains. If A and B are two observers on a space-like line (so that B is neither before nor after A), there are events (shaded in Fig. 3) which are common to A's future and B's past, and

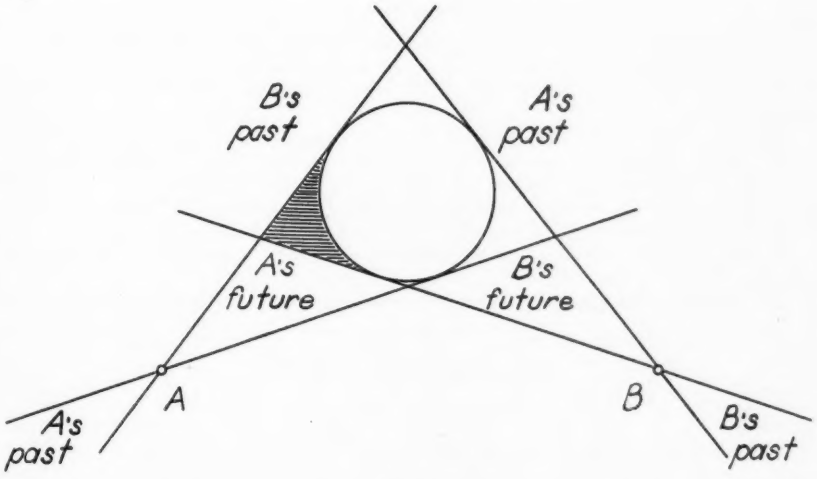

FIG. 3

other events common to A's past and B's future. This overlapping of the null cones becomes more pronounced as we increase the distance AB, until when $AB=\pi$ it is complete: A's future *is* B's past, and *vice versa*. But π (or πR) is the *whole* length of a space-like line, so A and B then coincide. In other words, if you could travel all along a space-like line, you would return to the starting-point with your past and future interchanged! There are two possible solutions for this paradox. You may argue that such a journey is essentially impossible:

the line joining you—"here and now"— to any event that you could possibly experience, is time-like. On the other hand, if you insist that the distinction between past and future is absolute, you can replace the exterior-hyperbolic space by its orientable covering manifold (which is representable on the whole one-sheeted hyper-sphere in Minkowskian space of 1+4 dimensions). But do you not find it disturbing to envisage an exact replica of yourself at the "antipodes," living backwards?

9. Conclusion. Apart from all these difficulties, we have ignored the embarrassing subject of the essential emptiness of de Sitter's world, which has led cosmologists to propose a modification in the direction of an earlier hypothesis: Einstein's "cylindrical" world. ([4], p. 160.) But the fact remains that de Sitter's is the theoretical world of greatest interest to pure geometers, as it alone has an interesting group. ([3], p. 657.) Since the points of exterior-hyperbolic space are the absolute poles of the hyperplanes of ordinary hyperbolic space (of four dimensions), the above remarks show that this group is the hyperbolic metric group: its elements are the collineations that preserve an oval quadric in real projective four-space.

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FOLDING THE CONICS

R. C. YATES, U. S. Military Academy

The idea of applying the process of paper folding to the construction of the conics originated with Row.* The methods he gives, however, are quite involved in a structural sense. Improvements (at least for the central conics) are offered by Lotka† but the usefulness and charm of his methods are somewhat obscured by an accompanying analytical justification.

The purpose of this note is to offer the best of the methods of Lotka and Row and to present simple proofs to establish the processes involved. It is hoped

* T. Sundara Row, Geometric Exercises in Paper Folding (trans. by Beman and Smith) Chicago, 1901.

† A. J. Lotka, School Science and Mathematics, VII, 1907, 595–597; Scientific American Supplement, 1912, 112.

that this formation of the conics by paper folding will somehow find its way into the clasroom where it will undoubtedly be received with enthusiasm.

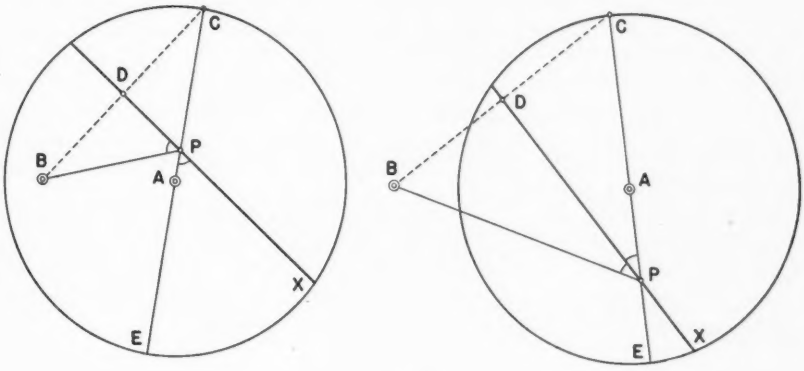

FIG. 1

A fixed point B is selected within a circle of radius r and center A. The point B is folded over upon the circle, as at C, forming the crease DX (the perpendicular bisector of BC) which meets the diameter EC in the point P. As B takes positions along the circle, the path of P is the *ellipse* having foci at A and B, major axis r, and the creases as tangents. For, since P lies on the perpendicular bisector of BC,

$$AP + BP = AP + PC = r$$

and

$$\angle BPD = \angle DPC = \angle APX.$$

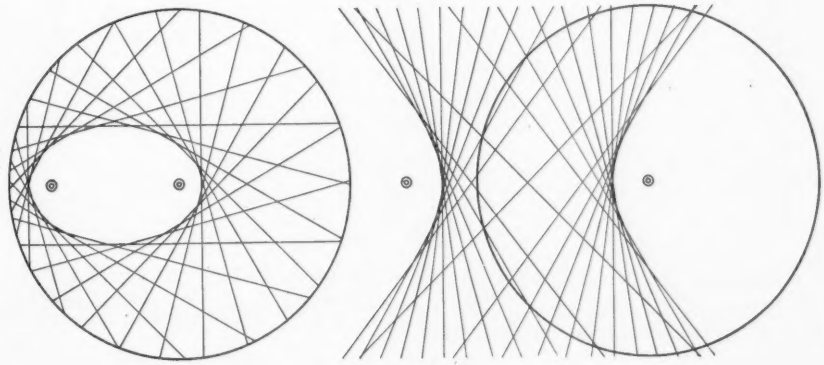

FIG. 2

(If B is taken at A, the locus of P is the circle with center at A and radius $r/2$.)

If B is selected outside of the circle, the locus of P (the intersection of crease and corresponding diameter) is the *hyperbola* having A and B as foci, real axis r, and the creases as tangents. For, since P lies on the perpendicular bisector of BC,

$$BP - AP = CP - AP = r$$

and

$$\angle BPD = \angle DPA.$$

The rectangular hyperbola is formed if $AB = r\sqrt{2}$. If B is taken on the circle, the locus of P is the point A.

The *parabola* (see Figure 3) presents a special case in which a line L replaces the circle of the central conics. The point B is folded over upon L, as at C, producing the crease DX. The perpendicular to L at C meets the crease in P. Points P form the parabola having B as focus, L as directrix, and the creases as tangents. For, since P lies on the perpendicular bisector of BC,

$$BP = PC$$

and

$$\angle BPD = \angle DPC.$$

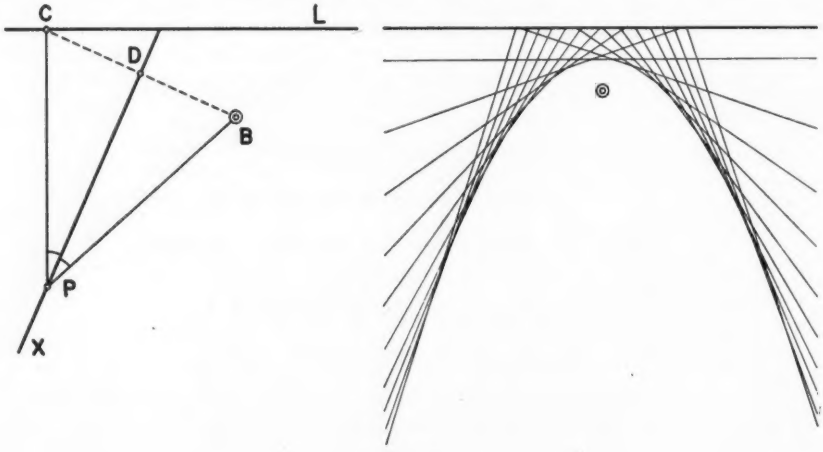

FIG. 3

This fascinating art of paper folding, which seems unfortunately relegated to the limbo, may be brought to full expression through the medium of ordinary wax paper found in every kitchen cabinet.

THE LIBRATION POINTS IN AN n-BODY PROBLEM*

J. J. L. HINRICHSEN, Iowa State College

1. Statement of problem. Let us consider the restricted problem of n bodies in which $n-1$ of the bodies have equal mass and are assumed fixed in the vertices of a regular $(n-1)$-sided polygon while the nth body is an infinitesimal mass, lying in the same plane, whose motion is influenced by the other $n-1$ bodies but which is unable to exert any influence upon them. The $n-1$ bodies are assumed to attract one another according to the Newtonian inverse square law and to be rotating in a circle about their common center of mass.

We shall choose the unit of mass as the sum of the masses of the $n-1$ bodies, the unit of distance as the distance from one of these bodies to the center of mass of the other $n-2$ bodies, and the unit of time such that the angular velocity of the system is unity.

We shall introduce a system of rectangular coordinates moving with the bodies such that the origin is at the center of the regular $(n-1)$-sided polygon. The x-axis shall be so chosen that for n even, one vertex of the polygon lies on the negative half of the x-axis and if n is odd, such vertices fall on both positive and negative sides of the axis. Then if the vertices of the regular polygon are denoted by P_i, their coordinates by (x_i, y_i), $(i=1, 2, \cdots, n-1)$, and the infinitesimal mass by P with coordinates (x, y), the potential function may be written

$$\Omega = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} \left[r_i^2 + \frac{2}{r_i} \right],$$

where

$$r_i^2 = (x - x_i)^2 + (y - y_i)^2 .$$

The equations of motion of the infinitesimal mass become

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = \Omega_x$$

$$\frac{d^2y}{dt^2} + 2\frac{dx}{dt} = \Omega_y,$$

where Ω_x and Ω_y denote the partial derivatives of Ω with respect to x and y respectively. The equations admit the well-known integral of Jacobi,

$$v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\Omega - C,$$

where C is a constant of integration and v denotes the velocity of P.

2. The libration points. This paper is concerned with the libration points of the system, namely those points whose coordinates satisfy

* Presented to the American Mathematical Society, November 21, 1941.

$$\Omega_x = \Omega_y = 0.$$

They are points about which infinitesimal periodic orbits may exist. They are also critical points of the system of curves

$$2\Omega - C = 0,$$

first discussed for the restricted problem of three bodies by G. W. Hill.*

We shall proceed to write the equations. If n is even,

$$x_i = r \cos \frac{2i-1}{n-1}\pi, \qquad y_i = r \sin \frac{2i-1}{n-1}\pi, \qquad (i = 1, 2, \cdots, n-1),$$

and if n is odd,

$$x_i = r \cos \frac{2i}{n-1}\pi, \qquad y_i = r \sin \frac{2i}{n-1}\pi, \qquad (i = 1, 2, \cdots, n-1),$$

where the distance from one body to the center of mass of the remaining $n-2$ bodies is set equal to unity. Then

$$r = \frac{n-2}{n-2+2\sum_{i=1}^{(n-2)/2} \cos \frac{2i-1}{n-1}\pi}, \quad \text{if } n \text{ is even, } (n > 2),$$

and

$$r = \frac{n-2}{n-1+2\sum_{i=1}^{(n-3)/2} \cos \frac{2i}{n-1}\pi}, \quad \text{if } n \text{ is odd, } (n > 3).$$

Now

$$\Omega_x = \sum_{i=1}^{n-1} \frac{\partial \Omega}{\partial r_i}\frac{\partial r_i}{\partial x} = \frac{1}{n-1}\sum_{i=1}^{n-1}\left(1 - \frac{1}{r_i^3}\right)(x - x_i),$$

$$\Omega_y = \sum_{i=1}^{n-1} \frac{\partial \Omega}{\partial r_i}\frac{\partial r_i}{\partial y} = \frac{1}{n-1}\sum_{i=1}^{n-1}\left(1 - \frac{1}{r_i^3}\right)(y - y_i),$$

and the equations whose solutions give the coordinates of the libration points may be written

$$\sum_{i=1}^{n-1}\left(1 - \frac{1}{r_i^3}\right)(x - x_i) = 0,$$

$$\sum_{i=1}^{n-1}\left(1 - \frac{1}{r_i^3}\right)(y - y_i) = 0.$$

* G. W. Hill, American Journal of Mathematics, vol. 1, 1878, pp. 5–29, 129–147, 245–260. Also see G. H. Darwin, Acta Mathematica, vol. 21, 1897, pp. 99–242.

If we multiply the first of these equations by $(y-y_j)$, the second by $(x-x_j)$ and subtract, we eliminate $(1-1/r_j^3)$ and obtain

$$\sum_{i=1}^{n-1} [(y_i - y_j)x - (x_i - x_j)y + (x_i y_j - x_j y_i)] \left(1 - \frac{1}{r_i^3}\right) = 0,$$

$$(j = 1, 2, \cdots, n-1).$$

These equations are not linearly independent. The term obtained for $i=j$ vanishes, and it is to be noted that

$$(y_i - y_j)x - (x_i - x_j)y - (x_i y_j - x_j y_i) = 0$$

is the equation of the straight line determined by the points (x_i, y_i), (x_j, y_j), while

$$1 - \frac{1}{r_i^3} = 0,$$

is the equation of the circle of radius one with (x_i, y_i) as center. These lines and circles divide the plane into regions. By investigating the signs of the terms of the above sums in a certain region, we are able to conclude the possibility or impossibility of existence of libration points in the region and are thereby able to establish certain bounds for those points. For $n=3$, 4 and 5, approximate locations of the libration points have been found.

3. The case $n=3$. This is the classical restricted problem of three bodies but with two equal masses. Here

$$x_1 = -\tfrac{1}{2}, \quad y_1 = 0; \qquad x_2 = +\tfrac{1}{2}, \quad y_2 = 0,$$

and the equations satisfied by the coordinates of the libration points are

$$y\left(1 - \frac{1}{r_1^3}\right) = 0$$

$$y\left(1 - \frac{1}{r_2^3}\right) = 0.$$

It follows either $y=0$ or $r_1=r_2=1$. In the first case, we may locate the libration points along the x-axis by approximating the roots of the equation $\Omega_x=0$, or

$$2x = \frac{x+\frac{1}{2}}{r_1^3} + \frac{x-\frac{1}{2}}{r_2^3}.$$

There exist five libration points of which three are collinear with the bodies and have coordinates $(\mu, 0)$, $(0, 0)$, $(-\mu, 0)$, where $1.20 < \mu < 1.21$ while the remaining two points $(0, \pm\sqrt{3}/2)$ lie at the vertices of equilateral triangles formed with the bodies as other vertices. The system of curves of zero velocity for a given energy constant C, studied by Hill, have these five points as critical points. Each of the libration points is within a distance $3/2$ from the center of mass of the system.

4. The case $n=4$. Our formulas yield

$$x_1 = \tfrac{1}{3}, \quad y_1 = \frac{1}{\sqrt{3}}; \qquad x_2 = -\tfrac{2}{3}, \quad y_2 = 0; \qquad x_3 = \tfrac{1}{3}, \quad y_3 = -\frac{1}{\sqrt{3}},$$

and the equations satisfied by the coordinates of the libration points are

$$\left(\frac{1}{\sqrt{3}}x - y + \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_2^3}\right) + \left(\frac{2}{\sqrt{3}}x - \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_3^3}\right) = 0,$$

$$\left(\frac{1}{\sqrt{3}}x - y + \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_1^3}\right) + \left(-\frac{1}{\sqrt{3}}x - y - \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_3^3}\right) = 0,$$

$$\left(\frac{2}{\sqrt{3}}x - \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_1^3}\right) + \left(\frac{1}{\sqrt{3}}x + y + \frac{2}{3\sqrt{3}}\right)\left(1 - \frac{1}{r_2^3}\right) = 0.$$

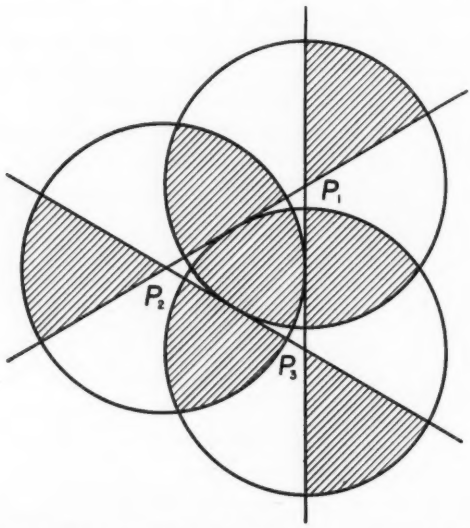

FIG. 1. Areas of the plane (shaded) in which Libration Points may lie for $n=4$.

The plane is divided into regions by the lines

$$\frac{1}{\sqrt{3}}x - y + \frac{2}{3\sqrt{3}} = 0,$$

$$\frac{2}{\sqrt{3}}x - \frac{2}{3\sqrt{3}} = 0,$$

$$-\frac{1}{\sqrt{3}}x - y - \frac{2}{3\sqrt{3}} = 0,$$

and the circles $r_i = 1$, $(i = 1, 2, 3)$. To be solutions of these equations, the coordinates (x, y) cannot make both terms of the first members either positive or negative. Such points can therefore only lie in the various shaded regions of the plane indicated in Figure 1.

Because of symmetry considerations, the libration points must lie along one of three axes of which $y = 0$ is typical. The position of the points along this line may be found by solving the equation $\Omega_x = 0$, or

$$3x = \frac{x - \frac{1}{3}}{r_1^3} + \frac{x + \frac{2}{3}}{r_2^3} + \frac{x - \frac{1}{3}}{r_3^3}.$$

There exist four roots which may be written $x = 0$, $\frac{1}{3}$, μ_1, and $-\mu_2$, where $.87 < \mu_1 < .88$ and $1.20 < \mu_2 < 1.21$. Hence there exists $1+3+3+3 = 10$ libration points in this restricted problem of four bodies.

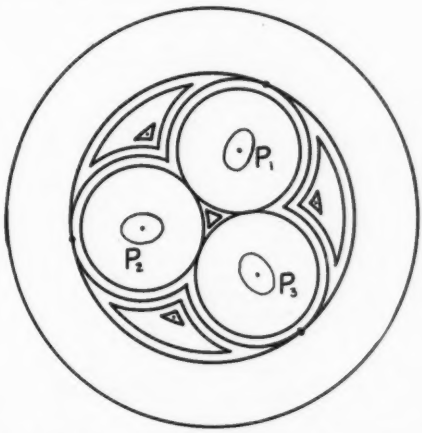

FIG. 2. Zero Velocity Curves for $n = 4$. (approximate)

5. Hill curves for $n = 4$. We shall next investigate the Hill curves for this problem and seek those points at which this system of curves has singularities. Consider the equation

$$2\Omega - C = 0,$$

or

$$r_1^2 + r_2^2 + r_3^2 + 2\left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}\right) = 3C.$$

For C large, either r_1, r_2 or r_3 must be large, in which case all three of these distances are large, or else one of the distances r_1, r_2 or r_3 must be very small. The motion must therefore be such that P is outside of a large closed curve enclosing the three bodies or else P is inside of a small closed curve about either P_1, P_2

or P_3. As C decreases the large curve diminishes in size while the small inner curves increase. The shapes of the curves also vary, but vary continuously with C. The first singularity arises when the three small curves simultaneously meet in three points on the sides of the equilateral triangle. This gives rise to three libration points namely those above corresponding to $x=\frac{1}{2}$. As C decreases further, the large outer curve continues to decrease while the three inner curves have now merged to form an outer and an inner closed curve. The outer one of these curves expands and meets the diminishing exterior curve simultaneously in three points namely those corresponding to $x=-\mu_2$. The innermost curve meanwhile decreases and shrinks to a point, disappearing in the origin $x=0$. The two outer curves, after meeting, break up into three separate closed curves each one of which shrinks to a point as C continues to decrease. These points correspond to $x=\mu_1$. Thus each one of the ten libration points is a critical point of the system of zero velocity curves for the problem under consideration. Each of the points is furthermore within a distance $5/3$ from the center of mass of the system.

6. The case $n=5$. The formulas yield

$$x_1=0,\quad y_1=\tfrac{3}{4};\qquad x_2=-\tfrac{3}{4},\quad y_2=0;\qquad x_3=0,\quad y_3=-\tfrac{3}{4},\quad x_4=\tfrac{3}{4},\quad y_4=0,$$

and the equations satisfied by the coordinates of the libration points are

$$(x-y+\tfrac{3}{4})\left(1-\frac{1}{r_2^3}\right)+2x\left(1-\frac{1}{r_3^3}\right)+(x+y-\tfrac{3}{4})\left(1-\frac{1}{r_4^3}\right)=0,$$

$$-(x-y+\tfrac{3}{4})\left(1-\frac{1}{r_1^3}\right)+(x+y+\tfrac{3}{4})\left(1-\frac{1}{r_3^3}\right)+2y\left(1-\frac{1}{r_4^3}\right)=0,$$

$$2x\left(1-\frac{1}{r_1^3}\right)+(x+y+\tfrac{3}{4})\left(1-\frac{1}{r_2^3}\right)+(x-y-\tfrac{3}{4})\left(1-\frac{1}{r_4^3}\right)=0,$$

$$(x+y-\tfrac{3}{4})\left(1-\frac{1}{r_1^3}\right)+2y\left(1-\frac{1}{r_2^3}\right)-(x-y-\tfrac{3}{4})\left(1-\frac{1}{r_3^3}\right)=0.$$

The plane is now divided into regions by the lines

$$x-y+\tfrac{3}{4}=0$$
$$x+y+\tfrac{3}{4}=0$$
$$x-y-\tfrac{3}{4}=0$$
$$x+y-\tfrac{3}{4}=0,$$
$$x=0$$
$$y=0,$$

and the circles $r_i=1$, $(i=1, 2, 3, 4)$. Each equation contains three terms in its first member and for their sum to vanish, these terms cannot all be either posi-

tive or negative. We are thus able to conclude that the libration points can lie only in certain restricted regions of the plane.

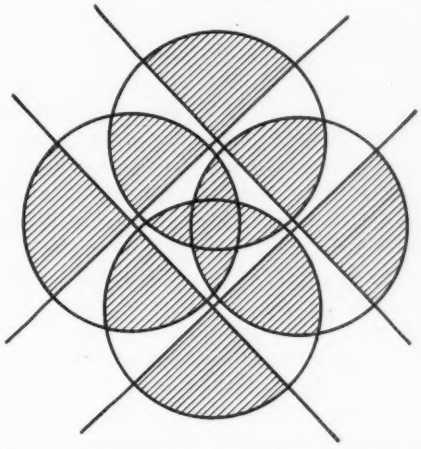

FIG. 3. Areas of the plane (shaded) in which Libration Points may lie for $n=5$.

Because of symmetry, the libration points must lie along one of four lines. Consider the lines $y=0$ and $y=x$ which are typical. The position of the points along $y=0$ may be found by solving the equation $\Omega_x=0$, or

$$4x=\frac{2x}{r_1^3}+\frac{x+\frac{3}{4}}{r_2^3}+\frac{x-\frac{3}{4}}{r_4^3}.$$

This equation possesses the roots $x=0$ and $x=\mu$, where $1.25<\mu<1.26$. For $y=x$, Ω_x becomes identical with Ω_y and we must solve

$$4x=\frac{2x-\frac{3}{4}}{r_1^3}+\frac{2x+\frac{3}{4}}{r_2^3},$$

which is found to have, in addition to $x=0$, two roots μ_1 and μ_2, where $.46<\mu_1<.47$ and $.62<\mu_2<.63$. With the symmetric points included, there exist $1+4+4+4=13$ libration points. Each is within a distance $7/4$ from the center of the system.

7. General remarks. In conclusion, we shall make several general remarks. There will exist $3n-2$ libration points each of which must lie within a distance $(2n-3)/(n-1)$ from the center of the system. As n increases, the regions within this distance excluded to libration points become smaller and smaller and have almost disappeared for $n=8$. The libration points must always lie on axes of symmetry. One will lie at the center of mass while $n-1$ will be equally distributed along each of three concentric circles about the center. The discussion of the system of curves of zero velocity may be extended to values of $n>3$.

MATHEMATICS AND MECHANICS IN THE POSTGRADUATE SCHOOL AT ANNAPOLIS*

R. E. ROOT, U. S. Naval Academy

1. The early years. Postgraduate education at Annapolis had its beginning in 1909, when a small group of young Naval Officers were ordered there to engage in studies in engineering science. For a few years this work proceeded without organized curricula, with no regular schedule of classes. A library was provided, and the facilities of the academic departments, and of the Engineering Experiment Station were available. The student officers were directed by a single officer, designated as the Head of the Postgraduate Department, who had the assistance of a civilian engineer and a small office force. The Department was housed in the loft of Isherwood Hall, the building occupied by the Department of Marine Engineering.

At this early period, as at all periods in the history of the school, the program benefited by generous cooperation of civilian institutions. The Head of the Department arranged for frequent lectures on engineering topics, some series of lectures constituting virtually a course of instruction. Professors from Columbia, Harvard, Johns Hopkins and perhaps other universities, and experts from the bureaus in Washington, participated in this program. Short courses, provided by part time assistance of members of academic departments, were increasingly relied upon to overcome deficiencies in the fundamental training of student officers. The deficiencies most evident were in mathematics, mechanics and physics. Mathematicians will be interested to know that Professor W. W. Johnson, then well past seventy, closed his illustrious career as a teacher by giving some of this part time instruction.

The outline of courses for 1914–15 contained courses in mathematics, mechanics, applied mechanics, physics, electrical engineering, machine design and thermodynamics. At this time the student body included only those taking mechanical and electrical engineering, and the year's work at Annapolis was preparatory to a year at Columbia University. The students were regular line officers of the Navy, with about five years of experience since graduation from the Naval Academy. The course in mathematics covered reviews in certain essentials of algebra, trigonometry and analytic geometry, becoming more thorough with the advanced topics in calculus, and including brief treatments of Fourier series, probability and precision, and ordinary differential equations. The course in mechanics covered statics and dynamics, and the applied mechanics covered strength of materials and the motion and balance of reciprocating engines. These three sequences were taught by one professor.

The work was interrupted by the declaration of war in April, 1917. Commander and student officers were at once ordered to other duty and the teaching force was assigned to the appropriate departments of the Naval Academy.

* Presented at the Annual meeting of the Mathematical Association of America, at Vassar College, September 7, 1942.

2. After the war. The school opened in its own building in June 1919, with Admiral E. J. King, then a Captain, as head of the department. Of the former teaching force, only two were on hand, the Professor of electrical engineering and physics and the Professor of mathematics and mechanics. A Professor of mechanical engineering and an assistant professor in each of the three divisions were provided at once. During the two years of Captain King's administration the organization was greatly improved. A professor in chemistry and metals was appointed and officer assistants were detailed to the school for administrative duties. The institution became officially the Postgraduate School. Curricula were established for mechanical, electrical, radio and aeronautical engineers, and in ordnance, naval construction and civil engineering. The first four curricula were identical for the year spent at Annapolis, specialization beginning the next year at other institutions. The other three groups could never quite be fitted into this system, but they did spend most of a year at Annapolis, and then go to other institutions.

The mathematics of this period was about the same as before the war, with the addition of a "mathematical laboratory" course of ten afternoons in the construction of charts and nomograms and in mechanical integration. The mechanics was expanded to include hydraulics and a smattering of aeronautics, and a course in modern ballistics was developed for the ordnance group.

During the next ten years, under the successors of Captain King, certain trends were evident and changes in fundamental policy were made. Emphasis on unification was limited to summer, fall and winter terms, and the spring term became a period of specialization, when each group was given courses especially planned to prepare for the next year's work at another institution. We were able to readjust our work in mechanics by having the hydraulics and strength of materials assigned to other divisions, reducing the general course to the conventional statics and dynamics, with a special course in engine balance and vibrations for mechanical and aeronautical engineers and a special course in aeronautics for aeronautical engineers. Ordnance students were assigned to sub-specialties, as torpedoes, explosives, metals, fire control, design and ballistics. All took much the same work at Annapolis, but went to different schools for one or two years of specialization. But after a few years it became evident that this degree of specialization was not desirable. Then the fire control, design and ballistics specialties were discontinued and replaced by a larger group in "general ordnance." This group took one full year at Annapolis, one summer at the Naval Gun Factory, then two more terms (eleven weeks per term) at Annapolis, followed by 14 months at various naval, army and industrial establishments. This second year ordnance curriculum called for advanced courses in differential equations, probability and statistical methods, dynamics of rigid bodies, exterior ballistics, and a course in applied elasticity emphasizing energy methods, dynamic stresses and vibrations.

During these years a General Line curriculum was introduced, for officers not specializing in any technical branch. In 1931–32 there were 63 officers in

this work, while 70 officers were in the first year technical curricula and 8 in second year ordnance. The general line courses included tactics, gunnery, communications, economics, government, international relations, administration and organization, psychology, navigation and electricity. Much of the teaching of these subjects was done by commissioned officers. In 1931 the Postgraduate School staff, under the Head of the School and the Executive officer, included 8 line officers and 15 civilian teachers, besides 2 visiting professors conducting certain general line courses. The civilians were divided, four in mathematics and mechanics, three in mechanical engineering, three in electrical engineering, two in metallurgy and chemistry, one in physics, one in radio, and one in modern languages.

3. The new plan. The year 1932–33 was a transition period, during which a new plan and revised curricula became effective. Under this plan all students were to enter the "School of the Line" for one year, selected groups were to follow specializing curricula the second year at Annapolis, after which qualified officers would take the advanced training at other institutions. The new class was to be divided at the beginning into groups corresponding to the various specialties, selection for certain sub-specialties to follow later.

For the pre-technical groups five hours per week were allotted to mathematics and mechanics, so that during the first year the mathematics could be covered up to differential equations, and the work in the regular statics and dynamics was finished the second summer. This permitted development of specialized courses in mathematics and mechanics for the various groups for the second year.

Under the new arrangement the naval constructors and civil engineers were not sent to Annapolis, but went directly to the Massachusetts Institute of Technology and to Rensselaer Polytechnic Institute. It should be understood that, ordinarily, these groups were selected within two years after graduation, usually from among those having very high scholastic standing, for transfer from the Line to the other Corps, while other groups were selected at least six years after graduation. The younger men could fit directly into the programs of civilian schools more easily than officers coming from six or seven years of sea duty. Moreover, the Construction Corps and the Civil Engineer Corps candidates would not require general line courses.

During the next few years the number of student-officers at Annapolis was sometimes as high as 275 men, over 100 of them in the second year work. The trend was away from unification, with a growing tendency to plan each curriculum to meet its particular objective, independent of other curricula. In most cases the students in a curriculum constituted at least one section suitable for separate class room instruction, and in few cases could the teaching load be effectively reduced by combining groups in different major curricula. Moreover, economy of teaching load is not a prime consideration when the students draw salaries comparable to those of the professors. Even if courses in different cur-

ricula were of similar content, differences in time schedules might make separate treatment necessary. This diversity of curricula was nearly complete for the second year students.

The general policy of the school, the character and objectives of the curricula and the assignment of student officers to these curricula, all are determined by the Head of School, in cooperation with the various bureaus of the Navy, and under the Bureau of Personnel, formerly the Bureau of Navigation. Each curriculum is administratively the charge of a commissioned officer who is a member of the staff. His responsibility extends to the welfare of the students in this specialty, and pertains to the work at other institutions as well as at Annapolis. Each of the courses designed to accomplish the general objective has a specific objective and a time allotment. It is the task of a professor of the subject to work out the details of a course, and we consider it well worth while to regard the content, arrangement and treatment of each course as a separate problem, to be considered in relation to the specific objectives and in coordination with other courses in the same curriculum.

Certain factors which affect the layout of courses are worthy of mention. First, the student body is very homogeneous, nearly all with a common Naval Academy experience followed by active naval duties. Occasionally, officers of the Marine Corps and Coast Guard take our courses, and a few officers from the navies of our South American neighbors are admitted, but in general we know what to expect. In fact, selections are so carefully made, usually after application of the candidate for the course, that failures in the courses are, indeed, very rare.

The second factor is the cooperation of the institutions to which our students are to go. Usually relations with these schools continue over many years, and mutual understanding eliminates unnecessary duplications and avoids serious omissions.

The third factor, particularly affecting revisions, is the comment of the students themselves after completing the courses, and sometimes after later naval experience. Often the officers in charge of curricula, and other officers of the staff, are former student officers.

With the increased divergence of curricula there was a growing tendency for the professors individually to center their interest in particular fields. Indeed this tendency had been encouraged from the start in relation to the specialized courses, by assigning each special course to the same teacher year after year. And now the attention of one professor in the mathematics and mechanics group is given almost entirely to aeronautical subjects, and that of another to ordnance. The remaining three of us have not been so fortunate as to confine our efforts to a single field, but each sequence of courses in a major curriculum has been made the particular responsibility of some one professor.

4. Courses in mathematics and mechanics. To this point I have spoken of conditions before the impact of the present national emergency on the work of

the school. I will now briefly characterize the courses given by the mathematics and mechanics professors in several major curricula just before this impact, say in 1939–40.

Naval Engineering. The objective here is to prepare officers to direct the design, inspection, operation and maintenance of naval machinery, and to direct related research.

The mathematics of the first year was followed by a course of 40 classroom hours in ordinary and partial differential equations, and by 40 hours divided among: (a) Numerical analysis, covering interpolation formulas, numerical integration and differentiation, numerical solution of algebraic and differential equations; (b) Solution by series, Bessel's equation and Bessel functions, and (c) introduction to vector analysis.

In mechanics the statics and dynamics were followed by 30 recitations and 10 problem periods, in advanced stress analysis, with emphasis on energy methods, and 40 recitations and 10 problem periods in hydromechanics. The problem periods are about two hours in length.

Radio Engineering. The objective is to prepare the officers to supervise the work relative to specification, design and research problems of radio and sound engineering, and to supervise the operation, maintenance and testing of radio and sound apparatus.

The first year mathematics was followed by a sequence of four courses in the four terms of the second year. (a) A course of 30 periods in ordinary differential equations, (b) 50 hours treating hyperbolic functions, harmonic analysis, numerical and series methods in differential equations, gamma and Bessel functions; (c) 50 hours in the application of differential equations to electric circuits, including Heaviside methods; and (d) 50 hours continuing Heaviside methods and including vector analysis with applications to electrical theory.

For this group the essential elements of mechanics were covered in physics courses.

Ordnance Engineering. The objective is to fit the officer to inspect ordnance material, to deal with problems of design and development in the Bureau of Ordnance, the Naval Gun Factory and the Naval Proving Ground, and to serve as experts in the operation of ordnance material in the fleet.

The mathematics of the second year was presented in four courses: (a) Twenty hours in ordinary differential equations; (b) 30 hours in statistical analysis including least squares, interpolation and frequency distributions; (c) 30 hours in exterior and interior ballistics, covering modern methods of trajectory computation and the derivation and application of the Le Duc formulas; and (d) 30 hours giving statistical methods covering correlation, theory of sampling, and control of quality by tests of samples.

The mechanics of the second year was in three courses: (a) Thirty class room hours and 10 problem periods in advanced stress analysis emphasizing energy methods; (b) 30 hours covering the theory of the thick cylinder with applications to built-up guns, also vector analysis with applications to dynamics of rigid

bodies; and (c) 30 hours covering the theory of the gyroscope and the stability of a rotating projectile in flight, also corrections of trajectories for non-standard conditions and for effects due to the earth's rotation, and the ballistics of bombing and of armor penetration.

Aeronautical Engineering. The objective is to fit the officer to cope with any problem arising in the naval aeronautical organization.

Three courses in mathematics in the second year were: (a) Thirty hours in ordinary differential equations; (b) 40 hours treating partial differential equations and numerical and series methods for ordinary differential equations, also vector analysis and complex variable theory applied to fluid mechanics; and (c) 30 recitations and 10 problem periods in aerodynamic theory of the airfoil.

The five courses in mechanics were: (a) Thirty class room periods and 10 problem periods in advanced stress analysis, including graphical and energy methods; (b) 30 hours and 10 problem periods in airplane structures; (c) 30 hours and 10 laboratory periods in aerodynamics of the airplane, including stability and performance and wind tunnel methods and equipment; (d) 40 hours and 10 problem periods, conducted in part as a seminar and dealing with problems of aircraft design; and (e) 30 hours and 10 problem periods in the dynamics of engine and shaft, analyzing periodic forces, balance of multicylinder engines, elastic vibrations and critical speeds.

Aerological Engineering. The objective is to prepare officers to become weather forecasters, to improve methods of forecasting at sea, and to participate in the solution of problems involving atmospheric conditions, as visibility, ballistic winds, etc.

Second year courses were: (a) Thirty hours in ordinary differential equations; (b) 40 hours in partial differential equations and vector analysis with applications in fluid mechanics; (c) 50 hours in mathematical processes in meteorology, deriving the general equations of motion on the rotating earth, and covering the kinematical methods of Petterssen for predicting changes in the pressure field; and (d) 40 hours in statistical analysis, covering interpolation, numerical differentiation and integration, numerical and series solutions of differential equations, least squares methods, correlation, Fourier analysis and the investigation of periodicity of data.

5. The present emergency. As I have said, when the war broke in 1917, the Postgraduate work was suspended. As the present national emergency approached, the school girded itself for greater burdens. Many months before Pearl Harbor the movement toward a two ocean navy had brought an urgent demand for many more naval officers with specialized technical training. The present Head of the Postgraduate School, Captain James A. Logan, realizing the important role that the school should play in the emergency, and foreseeing that the very urgent need for officers would make it difficult to have officers at school for long periods, directed revision of the curricula to reduce the time of each to the shortest minimum consistent with its objective.

General line subjects have been eliminated and some curricula of small tech-

nical content have been suspended, while the principal technical sequences have been shortened by months, and in some cases by years. Some emergency curricula have been added. With this adaptation of the school to the requirements of the emergency, the school is operating at full load, and only the limited capacity of the building prevents the ordering of larger groups to its courses.

I would mention two marked departures from previous conditions. First, the aerological course, which formerly took two years at Annapolis, and one year at another school, now is completed in five successive terms of ten weeks each, all at Annapolis. Officers who are competent forecasters conduct the practical part of the course, and the present class numbers 40, all but 5 of them reserve officers, graduates of universities or engineering schools, several with graduate training. Second, a curriculum for naval constructors has been established, running through three ten week terms, and when one class has finished another is ready to start. Each class, about 80 in number, is composed of young engineering graduates with commissions as reserve officers. They take a refresher course in mathematics, running five hours per week for ten weeks. The school calendar is now virtually five terms of ten weeks each.

Of the civilian teaching staff, most are reserve officers, and now only five of us are not in uniform. Of the force in mathematics and mechanics, formerly five in number, one has been ordered elsewhere, where his special abilities are required, and another divides the week between the Postgraduate School and the Naval Proving Ground at Dahlgren; but we have the assistance of four additional reserve officers, teachers from the departments of other institutions.

It will now be realized that the development of the Postgraduate School has not been hampered by any restricting conventions. As a school we have had no fixed standards to which we must conform other than those growing out of the needs of the navy. In our efforts to meet these needs we have made our own traditions.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Allegheny College, Meadville, Pa.

NUMERICAL INTEGRATION

J. S. FRAME

1. Introduction. The subject of numerical integration is one which could be of practical importance to students of engineering and other sciences if they but knew of it. However, it is usually not taught when the student begins integration, but is reserved for a course in advanced calculus which the majority of engineers and other scientists never take. The mathematics club can perform

a distinct service by bringing this topic to the attention of a wider group than the advanced mathematics majors.

The object of numerical integration is to obtain an approximate numerical value for a definite integral $\int_{x_0}^{x_0+l} f(x)dx$ in terms of certain known values of the integrand $f(x)$, and perhaps also in terms of additional data such as the values of the derivative $f'(x)$ at the ends of the interval. Assuming that the interval of integration (denoted by Δ), has finite length l, we denote the value of the integral by Al, so that A is by definition the *average value* of $f(x)$ on the interval Δ. Four different methods, or rules, for approximating this average A will be derived in this article. In each case an estimate will be given for the order of magnitude of the error. The comparative excellence of the approximations will then be illustrated by numerical examples. Finally, a brief mention of some applications will show instances when one of these methods gives exact results.

The function $f(x)$ will be assumed to be single-valued, continuous and integrable on the given interval Δ; and, in the last method of approximation, the derivative $f'(x)$ will also be assumed continuous.

2. Notation. We subdivide the given interval $\Delta: [x_0 \leqq x \leqq x_0+l]$ into n equal subintervals $\Delta_i: [x_{i-1} \leqq x \leqq x_i]$, whose equal lengths l/n are denoted by $2h$ and whose respective midpoints are denoted by a_i. We denote by b_i the ordinate at the midpoint a_i of Δ_i:

$$b_i = f(a_i) = f\left(\frac{x_{i-1} + x_i}{2}\right). \tag{2.1}$$

We further denote the ordinate at the right-hand endpoint x_i of Δ_i by y_i, the derivative $f'(x_i)$ by y_i', the average of the ordinates y_{i-1} and y_i by c_i, and the average of the extreme ordinates y_0 and y_n by C_1:

$$c_i = \tfrac{1}{2}(y_{i-1} + y_i) = \tfrac{1}{2}[f(x_{i-1}) + f(x_i)]. \tag{2.2}$$

$$C_1 = \tfrac{1}{2}(y_0 + y_n). \tag{2.3}$$

The symbol s_i will be used to denote that portion of the graph of the function $y=f(x)$ which corresponds to values of x in the interval Δ_i.

The average value of $f(x)$ on the interval Δ_i will be denoted by f_i. By our definitions this may be written in the two forms

$$f_i = \frac{n}{l}\int_{x_{i-1}}^{x_i} f(x)dx = \frac{1}{2h}\int_{a_i-h}^{a_i+h} f(x)dx. \tag{2.4}$$

The important quantity A, with whose value we are concerned, is easily seen to be the exact arithmetic mean of the n quantities f_i defined by (2.4).

$$A = (f_1 + f_2 + \cdots + f_n)/n. \tag{2.5}$$

Each of our approximations to A will be the arithmetic mean of n quantities which approximate f_i, so we shall have need of the two averages B_n and C_n, defined as follows:

$$B_n = (b_1 + b_2 + \cdots + b_n)/n, \tag{2.6}$$

$$C_n = (c_1 + c_2 + \cdots + c_n)/n. \tag{2.7}$$

By using (2.2) and (2.3) we may express C_n in the more useful form

$$C_n = (y_1 + y_2 + \cdots + y_{n-1} + C_1)/n. \tag{2.8}$$

3. Linear approximations: the tangent and trapezoid rules. Either b_i or c_i may be taken as an approximation to the average value f_i of $f(x)$ on Δ_i. In the first case the actual curve s_i is approximated by the tangent drawn at the extremity (a_i, b_i) of its mid-ordinate, and the area under the curve $y=f(x)$ is approximated by the "*tangent area*" under these segments of tangent lines, one of them drawn in each interval. Since the area under the ith tangent segment is $2hb_i$, the average ordinate under this segment is b_i. The arithmetic mean B_n of these mid-ordinates b_i is an approximation to the true average A.

$$\text{Tangent rule: } A \doteq B_n = \frac{\text{Area under tangents}}{\text{Length of interval}} = \text{Average of mid-ordinates of } n \text{ equal subintervals} \tag{3.1}$$

(The symbol $\doteq$ means "is approximately equal to").

In the second case, the area under the curve $y=f(x)$ is approximated in each interval Δ_i by the "*trapezoid area*" or "*chord area*" $2hc_i$ under the chord joining the end points of the arc s_i. We thus obtain as an approximation to A the arithmetic mean C_n of the n values c_i.

$$\text{Trapezoid rule: } A \doteq C_n = \frac{\text{Area under chords}}{\text{Length of interval}} = \frac{y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}(y_0 + y_n)}{n}. \tag{3.2}$$

4. Parabolic approximation: Simpson's rule. Instead of approximating the curve s_i by a line segment, we may approximate it by a parabolic arc through the three points (x_{i-1}, y_{i-1}), (a_i, b_i), (x_i, y_i). The computation is simplified if we introduce a new independent variable t:

$$t = (x - a_i)/h, \qquad -1 \leqq t \leqq 1, \tag{4.1}$$

in terms of which the equation of the parabola may be written

$$p_i(t) = b_i + \tfrac{1}{2}(y_i - y_{i-1})t + (c_i - b_i)t^2. \tag{4.2}$$

We verify by direct substitution, using (2.2), that this function takes on the values y_{i-1}, b_i, y_i at the points $t=-1$, 0, 1, respectively. We find by direct integration that the average ordinate p_i under this curve is as follows:

$$p_i = \frac{1}{2}\int_{-1}^{1} p_i(t)\,dt = b_i + 0 + \frac{1}{3}(c_i - b_i). \tag{4.3}$$

The arithmetic mean of the n quantities p_i, which we denote by P_n, is usually a much better approximation to A than either B_n or C_n. It is in fact a weighted average of B_n and C_n, which *gives the exact value of A whenever the function $f(x)$ is a polynomial of degree less than or equal to three.* The value of P_n may be expressed in several ways, using (4.3), (2.6), and (2.7).

$$\text{(4.4)} \qquad \textbf{Parabolic rule: } \mathbf{A \doteq P_n} = \frac{\text{Area under parabolic arcs}}{\text{Length of interval}} = \mathbf{B_n + \tfrac{1}{3}(C_n - B_n) = \tfrac{1}{3}(2B_n + C_n).}$$

This formula may also be written in a form known as Simpson's rule:

$$\text{(4.5)} \qquad \textit{Area} \doteq l(y_0 + 4b_1 + 2y_1 + 4b_2 + \cdots + 2y_{n-1} + 4b_n + y_n)/6n.$$

In a numerical example, however, it may save time to compute P_n (for $n>1$) from B_n and C_n as in (4.4), instead of using (4.5).

5. A quartic approximation. If the end values of the derivatives $f'(x)$ are known to the desired degree of accuracy (and if $f'(x)$ is assumed continuous), a still better approximation to A can be obtained with very little extra computation. We approximate the arc s_i by a quartic curve, passing through the three points (x_{i-1}, y_{i-1}), (a_i, b_i), (x_i, y_i) and also tangent to s_i at the first and third points. In terms of the variable t this polynomial $q_i(t)$ has the equation

$$\text{(5.1)} \qquad q_i(t) = p_i(t) + (t^3 - t)k_i + (t^4 - t^2)(b_i - c_i + d_i),$$

where

$$\text{(5.2)} \qquad d_i = 2h(y_i' - y_{i-1}')/8, \qquad k_i = (y_{i-1}'h + y_i'h + y_{i-1} - y_i)/4.$$

The function $q_i(t)$ is seen to coincide with $p_i(t)$ for $t=-1, 0, 1$. Furthermore, its derivative with respect to x, which is $q_i'(t)/h$, is seen to have the values y_{i-1}' and y_i' for $t=-1$ and 1, respectively. The average value q_i on the interval Δ_i is easily obtained by the integration of equation (5.1). It can be expressed as follows

$$\text{(5.3)} \qquad \begin{aligned} q_i &= p_i + 0 + (\tfrac{1}{5} - \tfrac{1}{3})(b_i - c_i + d_i), \\ &= b_i - \tfrac{1}{3}(b_i - c_i) - \tfrac{2}{15}(b_i - c_i + d_i). \end{aligned}$$

Fortunately, in forming the arithmetic mean D_n of the n quantities d_i, all the derivative values y_i' cancel out except the two end values $y_n' = f'(x_0 + l)$ and $y_0' = f'(x_0)$. For our new approximation to A the only new quantity to be computed in addition to B_n and C_n is D_n:

$$\text{(5.4)} \qquad D_n = [f'(x_0 + l) - f'(x_0)]l/8n^2 = (d_1 + d_2 + \cdots + d_n)/n.$$

We then obtain as a quartic approximation to A the arithmetic mean Q_n of the n quantities q_i, which is in many cases an even better estimate of A then is P_n.

In fact, it gives *A exactly whenever $f(x)$ is a polynomial of degree less than or equal to five.*

$$\text{(5.5)} \qquad \begin{aligned} \textbf{Quartic rule: } \mathbf{A} \doteq \mathbf{Q}_n &= \mathbf{P}_n - \tfrac{2}{15}(\mathbf{B}_n - \mathbf{C}_n + \mathbf{D}_n) \\ &= \mathbf{B}_n - \tfrac{1}{3}(\mathbf{B}_n - \mathbf{C}_n) - \tfrac{2}{15}(\mathbf{B}_n - \mathbf{C}_n + \mathbf{D}_n). \end{aligned}$$

6. Estimate of the error. In each of these methods of numerical integration, the error can be estimated if the derivatives of $f(x)$ are continuous and are known throughout the interval. Let M_r be the maximum value of $|f^{r}(x)|$ in Δ. Then the following inequalities can be shown to hold. We state them here without proof.

$$\text{(6.1)} \quad |A - B_n| < \frac{M_2}{24}\left(\frac{l}{n}\right)^2, \qquad \text{(6.2)} \quad |A - C_n| < \frac{M_2}{12}\left(\frac{l}{n}\right)^2,$$

$$\text{(6.3)} \quad |A - P_n| < \frac{M_4}{180}\left(\frac{l}{n}\right)^4, \qquad \text{(6.4)} \quad |A - Q_n| < \frac{M_6}{7!5!}\left(\frac{l}{n}\right)^6.$$

The errors obtained by use of the last two methods are seen to be small quantities of higher order in l/n than those of the first two. For a fixed value of n, however, it is possible to exhibit functions for which any chosen one of the four estimates B_n, C_n, P_n, Q_n is exactly equal to A, but the other three are different from A by as large an error as might be named. For instance, the quartic approximation is useless for a function whose derivative becomes infinite at one of the end points of the interval Δ, since D_n would then be infinite. In the examples which follow, however, it will be seen that Q_n gives the best approximation to A, and P_n the next best approximation.

7. Numerical examples. To illustrate the theory we shall compare the four approximations with the exact value of A for the two functions $1/(1+x^2)$ and $\sin x$ which can also be integrated by exact methods.

EXAMPLE 1. The integral $\int_0^1 (1+x^2)^{-1}dx$ has the value $\pi/4$, $=0.7853981634$ correct to ten decimal places. Since $l=1$, we have $A=\pi/4$. Let us first choose $n=1$, and then choose $n=5$, and estimate the integral by each of the four methods. For $n=1$ we have $D_1=-1/16=-.0625$. Furthermore,

$$\text{(7.1)} \qquad \begin{aligned} &B_1 = b_1 = f(\tfrac{1}{2}) = 0.8000, \qquad C_1 = \tfrac{1}{2}(1 + 0.5) = 0.7500, \\ &P_1 = B_1 - \tfrac{1}{3}(B_1 - C_1) = 0.7833, \qquad Q_1 = P_1 - \tfrac{2}{15}(-\ .0125) = 0.7850. \end{aligned}$$

Hence the errors in the four approximations are as follows:

$$\text{(7.2)} \quad A - B_1 = -0.0116, A - C_1 = 0.0354, A - P_1 = 0.0021, A - Q_1 = 0.0004.$$

The last method of approximation is obviously the most accurate in this case.

If we now take $n=5$ in the same example, we find that Q_5 is an approximation to $\pi/4$ which is accurate to eight decimal places. The computation can be arranged as follows.

$$f(x) = 1/(1+x^2), \quad 0 \leqq x \leqq 1, \quad n = 5, \quad h = 0.1.$$

$f(.1) =$	.99009 90099	$f(.2) =$	.96153 84615
$f(.3) =$	.91743 11927	$f(.4) =$	.86206 89655
$f(.5) =$	.80000 00000	$f(.6) =$	.73529 41176
$f(.7) =$	.67114 09396	$f(.8) =$	.60975 60976
$f(.9) =$	.55248 61878	$C_1 =$	.75000 00000
$5B_5 =$	3.93115 73300	$5C_5 =$	3.91865 76422

(7.3)

$B_5 =$	.78623 14660	$D_5 = -$	.00250 00000
$C_5 =$	.78373 15284	$B_5 - C_5 =$	.00249 99376
Divide by 3:		$B_5 - P_5 =$	.00083 33125
$P_5 =$	.78539 81535		
$Q_5 =$	.78539 81618	$A =$	.78539 81634 (by integration).

$-(B_5 - C_5 + D_5) =$ .00000 00624

.00000 00208

Sum/10 $= Q_5 - P_5 =$ .00000 00083

The errors in the four approximations are as follows:

$$A - B_5 = -.00083\ 33026, \quad A - C_5 = .00166\ 66350,$$
$$A - P_5 = .00000\ 00099, \quad A - Q_5 = .00000\ 00016. \tag{7.4}$$

As a check in the computation, it should be noticed that D_n is nearly equal to $C_n - B_n$ whenever the problem is such that the quartic approximation is a good one. If this condition is not satisfied, the computation should be checked to see if mistakes have been made.

EXAMPLE 2. Let us consider next the integral $\int_0^\pi \sin x\, dx$, whose exact value is 2. Taking $n = 3$, we can easily compute the required values of $f(x) = \sin x$ in terms of radicals. We have $l = \pi$, $A = 2/\pi$, $D_3 = -\pi/36$. Hence

$$3B_3 = (\tfrac{1}{2} + 1 + \tfrac{1}{2}) = 2, \quad 3C_3 = (\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\sqrt{3} + 0) = \sqrt{3}.$$

$B_3 =$	.66666 667	$D_3 = -$	.08726 646
$C_3 =$	.57735 027	$B_3 - C_3 =$	.08931 640
Divide by 3:		$B_3 - P_3 =$	.02977 213
$P_3 =$	.63689 453		
$Q_3 =$	.63662 121	$A =$	.63661 977 (by integration).

$B_3 - C_3 + D_3 =$ .00204 993

.00068 331

Sum/10 $= P_3 - Q_3 =$ .00027 332

The errors in the four approximations are as follows:

$$A - B_3 = -.03004\ 690 \quad A - C_3 = 0.5926\ 950$$
$$A - P_3 = -.00027\ 476 \quad A - Q_3 = -.00000\ 144. \tag{7.6}$$

In this particular example, it is easy to compare these errors with the estimates given in §6, since $M_2 = M_4 = M_6 = 1$, $l/n = \pi/3 = .10472$. These estimates are as follows:

$$|A - B_3| < (\pi/3)^2/24 = .045693, \quad |A - C_3| < .091386,$$
$$|A - P_3| < (\pi/3)^4/180 = .006681, \quad |A - Q_3| < .00000218. \tag{7.7}$$

It is obvious that in this example the quartic approximation is decidedly better than the other three.

8. Some exact averages. Problems in which the parabolic approximation P_1 gives the exact average A occur frequently in applications of integration. In the **prismoidal volume formula,** the volume of a solid is expressed as the product of the height by a certain average area of cross section:

(8.1) *Volume = Height (Lower base + 4 Mid-section + Upper base)/6.*

The average is precisely the average P_1, and the formula is exact for segments of such solids as cylinders, cones, spheres, wedges, prisms, and pyramids, for which the area of cross section is a polynomial of degree less than or equal to three in the distance from the lower base.

The problem of finding the moment of inertia of a straight rod or pipe of uniform density about an arbitrary axis in space is just that of integrating a squared distance r^2, which may be fairly complicated to express algebraically, but which is just a quadratic function of the arc length. The average value of r^2, which is called the square (k^2) of the radius of gyration, is given exactly by the parabolic approximation P_1, namely

$$k^2 = (r_0^2 + 4r_m^2 + r_1^2)/6, \tag{8.2}$$

where r_0, r_1, and r_m are the respective distances to the axis from the end points and midpoint.

DISCUSSIONS AND NOTES

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The department of Discussions and Notes is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A SIMPLE CRITERION FOR RATIONAL ROOTS

R. S. UNDERWOOD, Texas Technological College

This note calls attention to some tests for rational roots of algebraic equations which are simple and sweeping enough to be worthy of inclusion in any elementary algebra which has a chapter on the theory of equations.

Suppose c/d, $(c, d) = 1$, is indicated by the theorem on rational roots as a possible root of the algebraic equation with integral coefficients,

$$f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$$

With this notation, two theorems and a corollary may be stated briefly.

THEOREM 1. *We may reject c/d as a root of $f(x) = 0$ if $d - c$ is not a divisor of $f(1)$.*

This is a corollary of the general theorem given by L. E. Dickson in his *New First Course in the Theory of Equations*, page 29. It may be stated explicitly as follows: If c/d, $(c, d) = 1$, is a rational root of the algebraic equation $f(x) = 0$ with

integral coefficients, then $dm-c$ divides $f(m)$, m an integer.

COROLLARY. *We may reject c/d as a root of $f(x)=0$ if $f(1)$ is odd and $d-c$ is even.*

THEOREM 2. *If a_0, a_n, and $f(1)$ are odd, $f(x)=0$ has no rational roots.*

For, by the premises and the theorem on rational roots, c and d are odd, so that $d-c$ is even and the above Corollary applies.

In practice when Theorem 2 does not apply at once (and it evidently will apply only about one time in eight with a random choice of coefficients), the student should express $f(1)$ as the product of its prime factors. He can then use the Corollary and Theorem 1 in that order to reject by inspection and in wholesale lots many of the root-possibilities which he often tests laboriously via synthetic division.

A SUGGESTION FOR A SIMPLIFIED TRIGONOMETRY

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The Law of Cosines is not well adapted to the solution of an oblique triangle in which two sides and the included angle are given, and so the solution is usually accomplished by the use of the Law of Tangents. This latter procedure involves several auxiliary computations and is particularly undesirable in the case where the unknown angles are very nearly equal. There is, however, a little used third method which has some distinct advantages over the other two. It may be presented as follows.

We use the diagrams

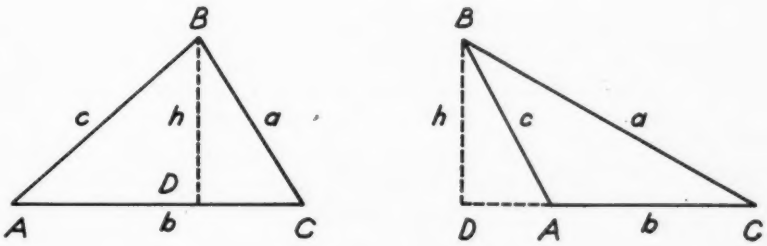

FIG. 1

In either figure

$$\cot A = \frac{AD}{h} = \frac{b - DC}{h} = \frac{b - a\cos C}{a\sin C}.$$

But then we have

$$\cot A = \frac{b}{a\sin C} - \cot C, \tag{1}$$

and the principal step in the solution has been accomplished. The other angle $B=180°-(A+C)$, but we may check the results by the use of the symmetrical formula

$$\cot B = \frac{a}{b \sin C} - \cot C. \tag{2}$$

The determination of A by the use of (1) requires the logarithmic computation of the quotient $b/(a \sin C)$ followed by two applications of a table of natural functions. The check formula requires relatively little additional tabular work. Formula (1) is comparable in derivation to the Law of Sines, and the corresponding computations begin with logarithmic work of precisely the same nature as in the Law of Sines.

Besides the application of (1) to triangle solution it may be combined with the Law of Sines to yield the formula for $\sin (A+C)$. It is customary to derive the latter formula by a complicated figure and only in the case where A, C, $A+C$ lie between 0° and 180°. Then A and C are the angles of a triangle in which $B=180°-(A+C)$, $\sin B=\sin (A+C)$. We use the Law of Sines to replace b/a by $\sin B/\sin A$ and have

$$\cot C + \cot A = \frac{\sin B}{\sin A \sin C}.$$

Solving for $\sin B$ we have

$$\sin (A + C) = \sin A \sin C \left(\frac{\cos C}{\sin C} + \frac{\cos A}{\sin A}\right) = \sin A \cos C + \cos A \sin C,$$

as desired. To derive the formula for $\cos (A+C) = -\cos B$ we use the symmetrical formula

$$\sin (A + B) = \sin C = \sin A \cos B + \cos A \sin B.$$

Then

$$\begin{aligned}\sin A \cos (A + C) &= \cos A \sin B - \sin C \\ &= \cos A(\sin A \cos C + \cos A \sin C) - \sin C(\sin^2 A + \cos^2 A) \\ &= \sin A(\cos A \cos C - \sin C \sin A)\end{aligned}$$

and we have the desired formula. The extension to general angles as usual is carried out analytically.

As a result of the procedure outlined above a simplified trigonometry may be constructed. The first four chapters on angular measurements, functions of a general angle, logarithms, functions of large angles might have the usual content. Chapter five would have as subject matter the solution of *all* triangles. It would contain formula (1) and the Law of Sines which would suffice for all cases of the oblique triangle save the relatively rare three sides case. The Law of Cosines

is well adapted to machine computation in this case and would be used.* Our final chapter would include all aspects of trigonometric analysis and would begin with the non-geometric derivation of the addition formulae presented above. This plan would seem to provide a more closely knit theory than any heretofore presented.

AN INSTRUMENT FOR DRAWING CONFOCAL ELLIPSES AND HYPERBOLAS†

R. M. SUTTON, Haverford College

In the physical problem of sound-ranging to find the location of a distant gun, the usual procedure consists in determining the time of arrival of sound

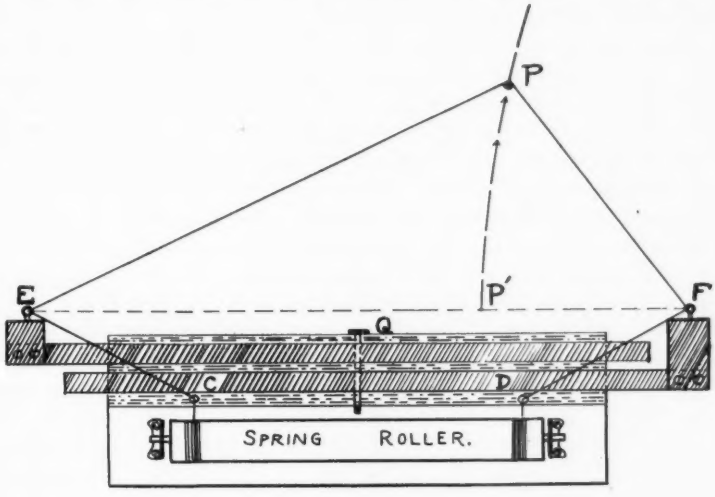

FIG. 1

from the gun at each of three or more stations. From the *difference* in time at any two stations, and from the known velocity of sound, it is possible to determine a hyperbolic curve with the two stations as foci, on which curve the source must lie; and if two or more such curves are determined, their intersection must coincide with the source of sound. To illustrate the manner of solving this problem graphically, the writer has made a simple instrument by which hyperbolas and ellipses may be drawn rapidly. As the idea of drawing such curves by string and chalk is very old, the merit of this arrangement must be sought, not in new principles, but in its convenience and flexibility.

On a plywood board $12'' \times 3''$ is mounted a window-shade spring roller with

* For use with tables of logarithms and natural functions it would be written in the form $\cos A + 1 = (a+b+c)(b+c-a)/2bc$.

† Shown at the Washington Meeting of the American Mathematical Society, May 3, 1941.

rachet removed (see figure) to which is tied an eight-foot length of fishline in two strands. If, now, the chalk is fixed to the line at any point P' and the line is pulled out, it pays out string at the same rate from the two sides of the roller through the small eyelets C, D, E, and F; the latter two eyelets are the foci of the hyperbola PP' described by P whose path has a constant *difference* of distance $(EP-PF)$ from the focal points E and F. It is a simple matter, with a little practice, to hold the chalk firmly against the cord and to move the point P out along its path of mechanical constraint under the tension of the roller. The eccentricity of the curve can be selected at will, and a whole family of confocal hyperbolas can be drawn in rapid succession by simply shifting the chalk on the string. Only one-half of each curve can be drawn at a time and the instrument must be rotated 180° to draw the second half.

For the drawing of ellipses, the roller can be held by the thumb of the left hand while the chalk, held in the right hand, traverses half of an ellipse as it slides along the cord. In this case, of course, the loop of cord is of fixed length during the drawing of any one ellipse, but the eccentricity of the ellipse may be changed readily by altering the length of loop within which the chalk is free to move. It is thus possible to draw in a few minutes a whole family of orthogonal confocal hyperbolas and ellipses. The distance between foci EF, may be changed quickly by removing a pin at Q so as to slide the two arms which bear the focal eyelets E and F in or out until the desired spacing is obtained. In the apparatus illustrated this distance can be altered from 6″ to 30″ in steps of one inch.

In physics, the rectangular hyperbola, $xy=k$, is of particular interest because many physical laws are reducible to equations of such a form. If one desires to graph the equation $xy=k$ on horizontal X- and vertical Y-axis, the instrument must be set at 45° to the X-axis and a family of curves all of the *same eccentricity* $(\sqrt{2})$ may then be generated by changing the focal distance each time a new member of the family is drawn. This is, however, a little more time-consuming than the process of drawing the confocal set previously described.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Mathematics for Technical Training, Calculus. By P. L. Evans. Boston, Ginn and Company, 1942. 7+126 pages. $1.25.

This book is the third in a series. It comprises both differential and integral calculus and their ordinary applications. As to pages it is indeed a brief course, but it is not abbreviated when it comes to the topics usually considered in a

standard course. There is no lack of problems, whether they be problems for guidance, worked out and explained, or the type of problems suitable for the home study of the reader, or for drill in the classroom. The interested student will be pleased to find that the answers are given for a sufficient number of the problems.

The text begins with a table of contents and closes with an index. Furthermore, the first chapter is a summary of formulas from the field of the prerequisite mathematics, and the last thirteen pages are given to a rather complete table of integrals. Principles are stated, but attempts at proofs, or derivations of the basic formulas are omitted. The book may also be considered as a suitable source for extra problems for the mastery of techniques. It is pleasing in its appearance and its arrangement.

H. T. R. AUDE

Mathematics of Modern Engineering, Vol. II. By E. G. Keller. New York, John Wiley and Sons, 1942. 12+309 pages. $4.00.

This book is one of a series appearing under the imprimatur of the General Electric Company. It is the second of two volumes bearing the same title and, according to its author, its aim is not so much to teach mathematics as to present to the practicing engineer various mathematical disciplines that are effective in the solution of certain engineering problems. Those of us who had to defend or explain the "practicality" of mathematics to engineering faculties and students cannot but feel grateful for the service being rendered by the author. The present book is divided into three broad chapters that may be labeled (the labeling is the reviewer's): Mathematico-Physical Principles; Matrices and Tensors; Differential Equations. The first chapter deals with the elements of the Calculus of Variations, Hamilton's principle, Lagrange's equations for holonomic and nonholonomic systems, Rayleigh's principle and classical vector analysis. The applications in this chapter are to dynamical systems and such problems as forced vibrations and oscillatory systems are treated.

The second chapter is devoted to matrices and tensors primarily as they are used in electrical networks. The definitions may seem unnecessarily restricted, but this may be justified by the objectives of the book.

The third chapter is concerned with the solution of systems of total differential equations, especially series solutions, their convergence and the various methods of approximate and numerical solutions. There is also a very brief account of elliptic and hyperelliptic functions and of Lalesco's non-linear integral equation. The applications are to electric circuits and rotating machines.

The reviewer feels that he would be derelict in his duty if he failed to point out many faults which the book possesses. The eternal mystery of the differential of a function is still unresolved in this book (p. 9) and even the definition of such a fundamental concept as potential energy is misstated (p. 19). One can hardly indorse "guessing" as one method suggested for solving an algebraic equation (p. 76) or "observing" that $x_2 = 4x_1/3$ minimizes w_a^2 and remarking that "This is a good estimate, since the exact value of x_2 is $4x_1/3$ (p. 87). It may be a matter

of taste that a great deal of space is devoted to dominant series and dismissing Picard's method by a trivial example without even mentioning the name, but one should object to equating (p. 233) the fourth approximation (a polynomial) to the solution (an exponential function).

Some of the "explanations" and definitions are either very much confused or are incorrect. There is no room—in the reviewer's opinion—in any book for such a statement as (p. 222). "We may think of a_i as constants, since they are the n arbitrary constants of the solution, or as variables which in turn have constant values for some specified values."

Many theorems or expressions which are valid only under certain restrictions are used without even mentioning what these restrictions are; nowhere in the book was the reviewer able to find the restriction necessary for the existence of the inverse of a matrix. Stating and using Cramer's rule without saying a word about the non-vanishing of the determinant of the coefficients is neither mathematics nor good engineering.

The reviewer found over forty misprints, and numerous statements, which, if not meaningless, are at least controversial. Many of the misprints are doubtless slips, but a number of them cannot be so explained. The whole section 2.25 is incorrect from its title to its emphasized statement (p. 127). The impedance tensor Z' on p. 151 is not the sum of the two tensors Z_1' and Z_2' as the sum is defined on p. 122, but the composition or direct sum of the two. The statement (p. 196) "shortest lines $\cdots$ in a curved affine space" is either nonsense or a misprint.

In spite of these blemishes, this book, as well as some of the others of this series, deserve wide circulation not only in industry but also in the engineering schools, and the reviewer hopes that books of this type—more carefully written—will become more frequently used.

M. S. KNEBELMAN

Calculus. By G. E. F. Sherwood and A. E. Taylor. New York, Prentice-Hall, Inc., 1942. 14+503 pages. $3.75.

This book is somewhat unusual in presenting a practical motivating "Foreword to the Student" which even precedes the Greek alphabet and the usual elementary formulas for reference.

The purpose of this text is "to set forth in systematic and thorough manner the fundamental principles, methods, and uses of the calculus."

If the engineering student can postpone one semester the use of integrals in his other courses he will find this an excellent text. The definite integral is started on p. 183 and precedes the formal treatment of indefinite integrals beginning in Chapter X on p. 198.

This text appeals to the reviewer as a most successful attempt to inject modern rigor into a practical first course for the sound training of the prospective engineer, mathematician, or scientist.

Limiting processes are emphasized from the start. Use is made of Cauchy's Principle of Convergence, uniform continuity, and Duhamel's Principle as enunciated by Osgood without mention of the word infinitesimal.

The authors are careful to state when assumptions are made and to state a theorem, even when the proof must be omitted, in order to secure accuracy of theory. For instance a careful distinction is made between double and iterated integrals. The Jacobian condition is stated in connection with implicit functions. Term by term integration is said not always to be valid for series which are not power series. Likewise term by term differentiation is not taken for granted even for power series.

Although the better students will appreciate the proofs of the theorems on limits of a sum, product and quotient, they may be omitted in a brief course, or if otherwise desired. The same holds true for several other sections such as those on evolutes, on roulettes, on the existence of an integral, on the osculating plane, and on envelopes.

As mentioned in the preface the chapters on hyperbolic functions, further methods of integration, infinite series, and Taylor's series, which occur rather late, might be taken up much earlier, just after Chapter X, or Chapter XI on geometrical applications. In fact part of Chapter XVI on hyperbolic functions could be studied just after Chapter V on differentiation of transcendental functions.

One valuable feature of the book is a thirty-three page chapter on solid analytic geometry. Another is the chapter summary outlining the important new ideas in each chapter and followed by a list of problems involving them. These lists are given in addition to the usual exercises under the separate topics. A majority of the answers are given but a sufficient number are omitted to invite the students to test their independence.

The format of the book is very pleasing. The paper is smooth, the print clear, and the figures are unusually good. There are over 138 numbered figures only a few of which might be improved, *e.g.*, 86a, 86b, 87a, 87b, on pp. 297, 298. A few defects in printing were noticed. On page 146 in the answer to problem 12 "A" is omitted; on p. 194 the fifth line from the bottom should end with -1; on p. 341, near the middle, the partial derivative of v lacks an "x"; on p. 500 under "indeterminate forms" 1° should be replaced by 1^∞. On p. 205, footnote, the statement that "log sin x has no meaning" is not quite accurate.

The book abounds in applications to Physics. Among these is an unusually full treatment of gravitational attraction. Another leads to the elliptic integral, which is handled in a practical way by infinite series.

One might have expected some discussion of the error in Simpson's rule. On the other hand one would not expect to find the infinite product for π derived from Wallis' integral formulas as given on page 382.

Preceding a good index of seven pages is a table of over a hundred integrals. The treatment of the limit e is properly made by considering the graph of

$(1+t)^{1/t}$ near $t=0$ and relegating an interesting existence proof to the back of the book as a four page appendix.

C. C. CAMP

Laboratory Geometry. By Elizabeth Roudebush. New York. Prentice-Hall, Inc., 1942. 192 pages. $1.12.

Laboratory Geometry is a combination text and work book designed for students at high school level. The book is a spiral note book, and therefore many of the exercises and constructions may easily be worked out in the spaces provided in the text. There are a great many illustrative figures; the print is large and very easy to read.

In Unit I the author has most successfully introduced the subject by the consideration of geometric figures: lines, angles, circles and triangles. The student's attention will be held by his active participation in supplying missing words and in performing simple constructions. The drawings are very instructive.

The idea of geometric proof, the axioms and postulates as well as the fundamental theorems on straight lines are included in Unit II. The proofs of the theorems are so well motivated that the student will be able to grasp the whole point of geometry more easily than he would, had the approach been purely axiomatic. Congruent triangles, polygons, and locus problems are dealt with in Units III, IV, and V respectively.

Excellent summaries at the end of each Unit list the definitions, axioms, postulates and theorems contained in the unit. A review of the first five units includes many practical problems. Circles, ratio and proportion, similar polygons, areas and regular polygons are treated in the Second Half of the book. In the unit on similar polygons a table of sines and tangents for degree intervals is included to enable the student to solve simple right triangles.

In my opinion the book is excellent both from the stand-point of teaching and material covered. Throughout, the ideas that are most difficult for the student to grasp are very carefully introduced in a conversational style by using worth while every day illustrations. The author is to be congratulated for the writing of Laboratory Geometry.

ELMER TOLSTED

NEW BOOKS RECEIVED

Mathematical Recreations. By M. Kraitchik. New York, W. W. Norton and Co., Inc., 1942. 328 pages. $3.75.

Seven place values of Trigonometric Functions for every thousandth of a degree. By J. Peters. New York, D. Van Nostrand Co., Inc., 1942. 376 pages. $7.50.

A Primer of Formal Logic. By J. C. Cooley. New York, Macmillan Co., 1942. 11+378 pages. $3.00.

Tables of arc tan x. New York, Work Projects Administration, 1942. 25+173 pages. $2.00.

Wartime Refresher in Fundamental Mathematics. By W. C. Eddy, A. H. Brolly, E. S. Pulliam, E. C. Upton, and G. W. Thomas. New York, Prentice-Hall, Inc., 1942. 8+248 pages. $1.40.

Solid Geometry. By A. M. Welchons, and W. R. Krickenberger. Revised Edition. Boston, Ginn and Co., 1943. 8+286 pages. $1.48.

Transients in Linear systems Studied by the Laplace Transformation. By M. F. Gardner and J. L. Barnes. Volume I. Lumped-Constant Systems. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1942. 9+389 pages. $5.00.

Introduction to Non-linear Mechanics. By N. Kryloff and N. Bogoliuboff. A free translation by Solomon Lefschetz of excerpts from two Russians monographs. (Annals of Mathematics Studies, no. 11.) Princeton University Press; London, Humphrey Milford, Oxford University Press, 1943. 5+109 pages. $1.65.

A Source Book of Mathematical Applications. By E. G. Olds, L. E. Boyer, R. E. Lane, N. W. Lazar and F. Lynwood. (The National Council of Teachers of Mathematics Seventeenth Yearbook.) New York, Teachers College, Columbia University, 1942. 16+291 pages. $2.00.

Spherical Trigonometry with Tables. By W. E. Brenke. New York, The Dryden Press, 1943. 8+27 pages. $0.80.

Popular Mathematics. By D. Miller. New York, Coward-McCann, Inc., 1942. 9+616 pages. $3.75.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 566. *Proposed by Michael Wilensky, Cincinnati, Ohio*

Suppose there is a principle according to which a claimant gets a share of the contestable thing, proportionate to his claim; so that when one of two claimants claims the whole, while the other claims only half of it, the former gets three quarters (*viz.*, the incontestable half, and half the contestable half) and the latter gets one quarter (which is half his claim). Find a formula for the share of each of n claimants, when the kth claimant claims $1/k$ of the entity $(k=1, 2, \cdots, n)$.

E 567. *Proposed by V. Thébault, San Sebastián, Spain*

Using compasses alone, construct a regular polygon of thirty sides.

E 568. *Proposed by P. D. Thomas, U. S. Coast and Geodetic Survey, Lucedale, Miss.*

In a given triangle show that the radical axes of the circumcircle with the respective circles whose diameters are any three concurrent Cevians meet the corresponding sides in three collinear points. (Cf. E 467 [1942, 63].)

E 569. *Proposed by David Matlack, Grinnell College*

Through a fixed point A on a circle (O), a line is drawn, parallel to a variable radius OP, meeting the circle again at Q. Find the envelope of the chord PQ.

E 570. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

If the six conics determined by each five of a set of six points are congruent, must they coincide?

SOLUTIONS

Circles through the Gergonne Point

E 527 [1942, 404]. *Proposed by V. Thébault, San Sebastián, Spain*

Show that the sum of the radii of the circles C_1, C_2, C_3 of E 457 is equal to the diameter of the incircle, and that the sum of the radii of the three analogous circles whose centers are exterior to the segments A_iI is three times as great.

Solution by Howard Eves, Syracuse University. Let us use the notation of E 457 [1941, 637], denoting the radius of C_i by r_i and the radius of the incircle by r. We shall use bars to designate the corresponding elements for the analogous set of circles.

Now we have $r_i/r = A_iP/A_iP_i$. But, from an elementary theorem on concurrency, $\sum A_iP/A_iP_i = 2$. Therefore $\sum r_i = 2r$.

Since C_iP and IP_i are corresponding lines of homothetic figures it follows that they are parallel, whence C_iP is perpendicular to A_jA_k. This property applied to C_j and C_k is sufficient to guarantee that A_{ji}, P, A_{ki} are collinear, whence $A_{ji}A_{ki}$ is the radical axis of C_i and $\overline{C}_i$. Therefore $A_{ji}A_{ij} = A_{ji}\overline{A}_{ij}$, or, since $A_{ji}P_k = P_kA_{ij}$, we have $P_k\overline{A}_{ij} = 3P_kA_{ij}$. From this it readily follows that $\bar{r}_i = 4r - 3r_i$, and finally $\sum \bar{r}_i = 6r$.

Note. If Q_i is the other point of intersection of A_iP_i with the incircle, we have incidentally shown that $\sum A_iP/A_iQ_i = 6$, and that the chords cut off on A_iP_i by C_i and $\overline{C}_i$ are quadrisected by Q_i and P_i, respectively.

Also solved by the proposer, using trigonometry.

An Orthocentric Group of Lines

E 533 [1942, 475]. *Proposed by N. A. Court, University of Oklahoma*

Prove that, if an orthocentric group of points occurs as a section of an orthocentric group of lines, then the plane of section is perpendicular to one of the lines.

Solution by the Proposer. Let A, B, C, H be the traces of the orthocentric group of lines DA, DB, DC, DH in a plane ABC. By assumption, the points A, B, C, H form an orthocentric group.

If the plane ABC is perpendicular to one of DA, DB, DC, the proposition is proved. We assume, therefore, that this is not the case, and shall show that the plane ABC is then necessarily perpendicular to DH.

If AA' is the perpendicular from A upon the plane DBC, and AA'' the perpendicular from A upon the line BC, then BC is perpendicular to the plane $AA'A''$. Now, since the four lines DA, DB, DC, DH are orthocentric, the plane DAH, perpendicular to DBC, contains AA'. Again, since the four points A, B, C, H are orthocentric, the perpendicular AA'' passes through H, and so lies in the plane DAH. Consequently the plane $AA'A''$ is identical with DAH. This plane is perpendicular to the line BC, and therefore to the plane ABC. In a like manner it may be shown that the planes DBH and DCH are each perpendicular to ABC. This proves, superabundantly, that DH is perpendicular to ABC.

Note. This is the converse of a known proposition. See the proposer's *Modern Pure Solid Geometry*, 1935, p. 28, art. 69.

An Unsolved Problem

E 534 [1942, 475]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Show that 4, 5, 7 are the only values of n for which $n!+1$ is a perfect square.

Remark by Paul Erdös, University of Pennsylvania. It would be very hard to prove even that $n!+1$ cannot be a perfect fourth power. In this direction, Obláth and I have proved the following results:

(1) The Diophantine equation $n!\pm m!=x^k$ has only a finite number of solutions if $n\geqq m>1$.

(2) The equation $n!=x^4\pm y^4$ has no solutions with n greater than a certain definite number.

(3) If x and y are relatively prime, $n!=x^k\pm y^k$ has no solutions for odd values of k, nor for $k=2^l$ where $l>2$.

Concurrent Simson Lines

E 535 [1942, 475]. *Proposed by A. H. Stone, Institute for Advanced Study*

Let A', B', C' be three points on the circumcircle of a triangle ABC, whose Simson lines with respect to ABC all meet in a point, O. Prove that the Simson lines of A, B, C, with respect to the triangle $A'B'C'$, concur at the same point O.

Solution by Howard Eves, Syracuse University. Let us designate the Simson lines with respect to ABC of A', B', C' by α', β', γ', and the Simson lines with respect to $A'B'C'$ of A, B, C by α, β, γ. We first establish a condition for α, β, γ to concur. The numbers in parentheses refer to articles in R. A. Johnson's *Modern Geometry*, 1929, where a treatment of this problem is outlined (338).

LEMMA. *A necessary and sufficient condition for α, β, γ to concur is that α be perpendicular to BC, β to CA, and γ to AB.*

(i) Suppose α, β, γ concur at a point O. Let H' be the orthocenter of $A'B'C'$, and produce $H'O$ its own length to H. Now, since α bisects both AH' (327) and HH', it follows that α is parallel to AH. Similarly β is parallel to BH, and γ to CH. Therefore

$$\sphericalangle BHC = \sphericalangle(\beta\gamma) = \sphericalangle CAB$$

(326, second corollary). (We are here using Johnson's *directed angles* (16). In the ordinary notation $\angle BHC = 180° - A$, with a suitable modification if B or C happens to be obtuse.) Similarly we have

$$\sphericalangle CHA = \sphericalangle ABC \quad \text{and} \quad \sphericalangle AHB = \sphericalangle BCA.$$

Hence H is the orthocenter of ABC. Since α is parallel to AH, it then follows that α is perpendicular to BC, with similar remarks for β and γ.

(ii) Suppose α is perpendicular to BC, β to CA, γ to AB. Let H be the orthocenter of ABC, and let O be the midpoint of HH'. Now α is parallel to AH and bisects AH'. Therefore α passes through O. Similarly β and γ pass through O. Thus α, β, γ concur.

We use the above lemma to show that the concurrence of α', β', γ' implies the concurrence of α, β, γ (at the same point O). Let the isogonal of $A'A$ with respect to angle A' be $A'A''$ (with A'' on the circumcircle). Then α is perpendicular to $A'A''$ (326, Theorem). It is readily seen that AA'' is parallel to $B'C'$, whence by the lemma, AA'' is perpendicular to α'. This guarantees that AA'' and AA' are isogonals with respect to angle A, so that $A'A''$ is parallel to BC, whence α is perpendicular to BC. Similarly β is perpendicular to CA, and γ to AB. This proves that α, β, γ concur. As in (ii), then, it follows that the point of concurrence is the midpoint of HH', where H and H' are the orthocenters of ABC and $A'B'C'$.

A Needle in a Bowl

E 536 [1942, 546]. *Proposed by Norman Miller, Queen's University*

In a smooth hemispherical bowl of radius a, a smooth needle of length $2l$ ($l<a$) is placed with one end projecting over the rim and is then released. Show that the needle will come to rest in a horizontal position if l is less than $a\sqrt{\frac{2}{3}}$.

Solution by Howard Eves, Syracuse University. Let A and B be the ends of the needle, M its midpoint, and O the center of the bowl. Three situations are possible when the needle comes to rest: (1) A and B may both be within the bowl; (2) A, say, may be within the bowl while B projects outside the rim; (3) A may be within the bowl and B just on the rim.

In (1) the needle is acted on by three forces: those at A and B, each acting normal to the bowl (and hence through O), and one acting vertically at M. Since the needle is in equilibrium, these three forces are concurrent. But for the vertical force at M to pass through O we must have the needle horizontal.

In (2) the needle is again acted on by three forces: one at A, normal to the bowl, a second at C (where the needle touches the rim) normal to the needle, and a third acting vertically at M. Again the three forces must be concurrent, and from the geometry of the situation the point of concurrence is D, diametrically opposite to A on the sphere determined by the bowl. Here, of course, the needle is inclined.

In (3) the needle is in unstable equilibrium.

To find the condition on l leading to the three cases, set $MC=m<l$, $DC=n$, $\angle BAD=\angle CDM=\theta$. Then

$$n/(l+m) = \tan\theta = m/n,$$

whence $n^2=(l+mm)$. Now, by Pythagoras' theorem, we have

$$4a^2 = (l+m)^2 + n^2 = (l+m)(l+2m) < 6l^2,$$

whence $l>a\sqrt{\frac{2}{3}}$. This is the condition for situation (2). If $l=a\sqrt{\frac{2}{3}}$ we have (3), and if $l<a\sqrt{\frac{2}{3}}$ we have (1).

For an alternative method of approach see Bowser's *Analytic Mechanics*, p. 82, Ex. 9.

Remark by W. B. Carver, Cornell University. Suppose l lies between $a\sqrt{\frac{2}{3}}$ and a. If the needle is released from a sufficiently inclined position, it will slide past the position of equilibrium with sufficient momentum to slip entirely into the bowl, and will come to rest in the horizontal position, as in case (1).

Also solved by J. H. Butchart, L. M. Kelly, W. O. Pennell, K. Yamakawa, and the proposer.

Editorial Note. In view of Professor Carver's remark, the wording of the problem has been slightly changed since its first appearance.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4078. *Proposed by Raphael Robinson, Univ. of California at Berkeley*

Show that the continued fraction

$$\frac{1}{2^{1!}} + \frac{1}{2^{2!}} + \frac{1}{2^{3!}} + \frac{1}{2^{4!}} + \cdots,$$

represents a transcendental number by using the following theorem of Liouville, *Journal de Mathématiques pures et appliquées*, 1851.

The denominator of a convergent of a continued fraction representing the root x of an nth degree algebraic equation with rational coefficients never exceed the product of a certain constant by the $(n-2)$nd power of the denominator of the preceding convergent.

4079. *Proposed by S. Beatty, Univ. of Toronto*

If two complexes, in projective n-space, are reciprocal with respect to a quadric, so that each vertex of one is the pole of a bounding hyperplane of the other, then the $n+1$ lines joining corresponding vertices are associated,* in the sense that every $(n-2)$-space which meets n of them meets the remaining one also. Reciprocally, the $n+1$ of the $(n-2)$-spaces of intersection of corresponding hyperplanes are such that every line which meets n of them meets the remaining one also.

4080. *Proposed by V. Thébault, San Sebastián, Spain*

Given five points in space, if five parallel forces applied at these points are in equilibrium, then equilibrium results also when each force is translated to the center of the circumsphere of the tetrahedron which has for vertices the points of application of the four other forces.

Note. The similar problem for four points in a plane was proposed by J. Neuberg, *Educational Times*, 1891, vol. 55, p. 82.

4081. *Proposed by Otto Dunkel, Washington Univ.*

Through the vertices of the triangle ABC parallels $A\alpha$, $B\beta$, $C\gamma$ of arbitrary direction are drawn meeting the transversal Δ in the points α, β, γ; and through the latter points straight lines are drawn parallel respectively to BC, CA, AB rotated through the angle θ, thus forming a triangle $A_1B_1C_1$ similar to ABC. Prove that: (1) As the direction of the parallels varies the vertices A_1, B_1, C_1 describe straight lines concurrent in a point $\phi(\theta)$. (2) For the particular set of parallels $A\alpha$, $B\beta$, $C\gamma$ which have the direction of Δ rotated through the angle $-\theta$, the triangle $A_1B_1C_1$ reduces to the point $\phi(\theta)$. (3) The locus of $\phi(\theta)$ is a unicursal cubic passing through the circular points at infinity, the point at infinity of the Newton line (ABC, Δ), the orthopole of Δ with respect to ABC, and A_0, B_0, C_0, the points of intersection of the sides of ABC with Δ.

SOLUTIONS

Miquel Circle and Point

3890 [1938, 554]. *Proposed by V. Thébault, San Sebastián, Spain*

Given four straight lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, in a plane; through the orthogonal projections of the vertices of the triangles $T_1 \equiv (\Delta_2, \Delta_3, \Delta_4)$, $T_2 \equiv (\Delta_1, \Delta_3, \Delta_4)$, $T_3 \equiv (\Delta_1, \Delta_2, \Delta_4)$, $T_4 \equiv (\Delta_1, \Delta_2, \Delta_3)$ on $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, respectively, parallels are

* For example, when $n=3$, the four lines are generators of a regulus. The case where $n=2$ is Hesse's theorem.

drawn to the sides opposite the corresponding vertices of the triangle considered: these parallels determine four other triangles T_1', T_2', T_3', T_4' symmetrically equal to the first. (1) Show that the Miquel circles of the quadrilaterals (T_1', Δ_1), (T_2', Δ_2), (T_3', Δ_3), (T_4', Δ_4) are equal to the nine-point circles of T_1, T_2, T_3, T_4 and tangent to the circumcircles of T_1', T_2', T_3', T_4'. (2) Show that the Miquel points of the above quadrilaterals are collinear.

Solution by the Proposer. (1) The first part of this problem is a direct application of the following:

THEOREM. *The parallels to the sides BC, CA, AB of a triangle ABC drawn through the orthogonal projections α, β, γ of the vertices A, B, C on any given straight line Δ of the same plane determine a triangle $A_1B_1C_1$ symmetrically equal to ABC. The Miquel circle of the quadrilateral $[A_1B_1C_1, \Delta]$ is equal to the nine point circle of ABC and it is tangent to the circumcircle of $A_1B_1C_1$ at the orthopole ϕ of Δ with respect to ABC. Also the respective symmetrics of the vertices A_1, B_1, C_1 with respect to the circumcenters of triangles $A_1\beta\gamma$, $B_1\gamma\alpha$, $C_1\alpha\beta$ coincide with the point ϕ.* (V. Thébault, *Mathesis*, vol. 49, 1935, p. 298.)

Proof. Suppose first that Δ' is a diameter of the circumcircle (O) of triangle ABC parallel to Δ. The parallels to the sides BC, CA, AB drawn through the orthogonal projections α', β', γ' of the vertices A, B, C on Δ' determine a triangle $a_1b_1c_1$ symmetrically equal to triangle ABC.* We have also shown that the orthopole ϕ' of the diameter Δ' for triangle ABC is the common symmetric of the points a_1, b_1, c_1 with respect to the midpoints A_m, B_m, C_m of the sides of triangle ABC, and that it is the point of contact of the circumcircle (O_1) of $a_1b_1c_1$ with the nine point circle (O_9) of ABC.† From this we infer that the circle (O_9) is the Miquel circle for the quadrilateral $[a_1b_1c_1, \Delta']$. If we now translate Δ' with the attached triangle $a_1b_1c_1$ by the vector δ perpendicular to it to the parallel position Δ, the triangle $a_1b_1c_1$ assuming the position $A_1B_1C_1$, the latter triangle has its sides parallel to the corresponding sides of ABC and passing through the translated points α, β, γ which are the orthogonal projections of A, B, C on Δ, and $A_1B_1C_1$ is symmetrically equal to ABC.‡

The Simson line with respect to $A_1B_1C_1$ of the orthopole ϕ of the line Δ for the triangle ABC coincides with Δ; if σ is the Miquel circle for the quadrilateral $[A_1B_1C_1, \Delta]$ the figure $\{A_1B_1C_1, \sigma\}$ is the translation of $\{a_1b_1c_1, (O_9)\}$ by the translation δ, and the theorem is proved. It follows that the point ϕ on the circumcircle of $A_1B_1C_1$§ is the focus of a parabola inscribed in this triangle with Δ as the tangent at the parabola vertex. Thus the point ϕ is the Miquel point of the complete quadrilateral $[A_1B_1C_1, \Delta]$. The proof of the first part of the problem is now complete.

(2) The second part follows from the theorem that the orthopoles ϕ_1, ϕ_2, ϕ_3, ϕ_4

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* J. Neuberg, *Wiskundig Tydschrift*, vol. 10, p. 80.

† V. Thébault, *Mathesis*, vol. 49, 1935, p. 270.

‡ J. Neuberg, *loc. cit.*

§ V. Thébault, *Nouvelles Annales de Mathématiques*, 1914, p. 223.

of the straight lines $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ with respect to the triangles T_1, T_2, T_3, T_4 are collinear* and coincide as shown above with the Miquel points of the quadrilaterals $[T_1', \Delta_1]$, $[T_2', \Delta_2]$, $[T_3', \Delta_3]$, $[T_4', \Delta_4]$.

Note. The parallels to $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ drawn through the orthocenters H_1, H_2, H_3, H_4 of triangles T_1, T_2, T_3, T_4 form a complete quadrilateral such that the four triangles $T_1'', T_2'', T_3'', T_4''$ are symmetrically equal to the triangles T_1, T_2, T_3, T_4. The orthopole ψ_1 of the straight line $\Delta \equiv (H_1, H_2, H_3, H_4)$ for the triangle T_1 lies on a parallel Δ'' to Δ symmetric with respect to Δ to the tangent Δ' at the vertex of the parabola with focus F inscribed in the quadrilateral $(Q) \equiv [\Delta_1, \Delta_2, \Delta_3, \Delta_4]$. The orthopole ψ_1' of Δ with respect to triangle T_1'' is therefore on the straight line symmetric to Δ'' with respect to Δ, that is on Δ', for the center of symmetry of triangles T_1 and T_1'' is the intersection of Δ with the Newton line of the quadrilateral (Q),† and moreover we know that the orthopoles of a line Δ with respect to two homothetic triangles T_1 and T_1'' are collinear with the projection on the considered straight line of the homothetic center σ of the two triangles, and their distances from σ are in the ratio of the homothetic ratio of the two triangles.‡ Similarly, the orthopoles $\psi_2'', \psi_3'', \psi_4''$ of Δ in the triangles T_2'', T_3'', T_4'' are also on Δ'. But we know also that $\psi_1'', \psi_2'', \psi_3'', \psi_4''$ lie on the orthocentric lines of the quadrilaterals $[T_1'', \Delta]$, $[T_2', \Delta]$, $[T_3'', \Delta]$, $[T_4'', \Delta]$, that is on $\Delta_1, \Delta_2, \Delta_3, \Delta_4$.§ From this follows:

THEOREM. *The orthopoles $\psi_1'', \psi_2'', \psi_3'', \psi_4''$ of the orthocentric line Δ of the quadrilateral (Q) with respect to triangles $T_1'', T_2'', T_3'', T_4''$ coincide with the orthogonal projections of the Miquel point F on the sides of the quadrilateral (Q).*

This property is also true for the orthopoles $\psi_1', \psi_2', \psi_3', \psi_4'$ of the straight line $\Delta \equiv (H_1, H_2, H_3, H_4)$ with respect to the triangles T_1', T_2', T_3', T_4', and in general for the figure where we consider the variable parallels projecting the vertices of the triangles T_1, T_2, T_3, T_4 on $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. For the triangles T_1', T_2', T_3', T_4', and in general $T_1^i, T_2^i, T_3^i, T_4^i$, are obtained from $T_1'', T_2'', T_3'', T_4''$ by translations perpendicular to Δ. The orthopoles $\psi_1'', \psi_2'', \psi_3'', \psi_4''$ remain fixed, coinciding therefore with those for the line Δ for all of these quadruples of triangles.

Editorial Note. It is assumed here that no two of the lines Δ_i are parallel and that their six intersections A_{jk} are distinct. The theorem in the first part of the above solution results from the following considerations. Let parallel straight lines of arbitrary direction through A and B of triangle ABC meet the transversal Δ in α and β, and let parallels to BC and CA through α and β respectively intersect in C_1. Consider the six straight lines $CA, AB, BC, C_1\alpha, \Delta = \alpha\beta, \beta C_1$, and for simplicity suppose $\alpha \neq \beta$. The intersections of the pairs of parallels $CA, \beta C_1$ and $BC, C_1\alpha$ determine the line at infinity, and $A\alpha$ and $B\beta$ meet on this line.

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* R. Goormaghtigh, *Nouvelles Annales de Mathématiques*, 1919, p. 39.

† V. Thébault, *Mathesis*, 1937, pp. 187–242.

‡ V. Thébault, *Mathesis*, 1933 (supplement, p. 31).

§ R. Goormaghtigh, *Nouvelles Annales*, 1939, p. 39, and V. Thébault *Mathesis*, 1937, *loc. cit.*

From this it follows that a conic $\{S\}$ is tangent to these six lines, and its center S is the midpoint of CC_1. Hence the symmetric of AB with respect to S is also a tangent forming with the two lines through C_1 the triangle $A_1B_1C_1$ symmetrically equal to ABC with respect to S. Let A_1B_1 meet Δ in γ and consider the six tangents AB, BC, CA, $A_1\beta$, $\beta\gamma$, γA_1. Here the parallel pairs CA, $A_1\beta$ and γA_1, AB determine the line at infinity, and hence $B\beta$, $C\gamma$ are parallel.

Let Δ meet BC, CA, AB in A_0, B_0, C_0, then taking first the six tangents C_1A_1, A_1B_1, B_1C_1, CA_0, A_0B_0, B_0C, and next the six A_1B_1, B_1C_1, C_1A_1, AB_0, B_0C_0, C_0A, we show that A_0A_1, B_0B_1, C_0C_1 are parallel. The three triangles AA_0A_1, BB_0B_1, CC_0C_1 are such that the common midpoint of AA_1, BB_1, CC_1 is S, and the straight lines from S to the midpoints of AA_0, BB_0, CC_0 are parallel respectively to the parallels A_0A_1, B_0B_1, C_0C_1. Hence S and the midpoints of AA_0, BB_0, CC_0 are collinear on (ABC, Δ), and by similar reasoning we show that S lies also on $(A_1B_1C_1, \Delta)$. From this it follows that the parabola tangent to Δ and to the sides of ABC has its axis parallel to (ABC, Δ); see the solutions of 3817 [1939, 177] and 3818 [1939, 178].

Suppose now that the parallels $A\alpha$, $B\beta$, $C\gamma$ are perpendicular to Δ, then it is known that the perpendiculars from α, β, γ to BC, or B_1C_1, *etc.*, meet in a point ϕ, the orthopole of Δ with respect to ABC. From the fact that the projections α, β, γ of ϕ on the sides of $A_1B_1C_1$ lie on the straight line Δ, it is also known that ϕ lies on the circumcircle (O_4) of $A_1B_1C_1$. Since $A_1\gamma\phi$ and $A_1\beta\phi$ are right triangles the circumcircle (O_1) of triangle $A_1\gamma\beta$ has its center O_1 at the midpoint of ϕA_1; and since ϕA_1 is the common chord of (O_1) and (O_4) the angle ϕO_1O_4 is a right angle. If O_2 and O_3 are the circumcenters of $B_1\alpha\gamma$ and $C_1\beta\alpha$, we see similarly that ϕO_2O_4 and ϕO_3O_4 are also right angles. Hence O_1, O_2, O_3, O_4, ϕ are on a circle (M) with its center M at the midpoint of ϕO_4; and (O_4) and (M) are tangent at ϕ. In this special case $(A_1B_1C_1, \Delta)$ is also perpendicular to Δ. It follows from the above that ϕ is the focus of a parabola tangent to the sides of $A_1B_1C_1$ with Δ as the vertex tangent. It also follows that, if parallel straight lines of arbitrary direction through A_1, B_1, C_1 meet Δ in $\alpha_i, \beta_i, \gamma_i$, the parallels to B_1C_1, C_1A_1, A_1B_1 through the latter points form a triangle $A_iB_iC_i$ symmetrically equal to $A_1B_1C_1$ with the center of symmetry S_i on $(A_1B_1C_1, \Delta)$, and $A_i\alpha$, $B_i\beta$, $C_i\gamma$ are parallel to $(A_1B_1C_1, \Delta)$ and consequently perpendicular to Δ. Thus ϕ is the orthopole of Δ for each $A_iB_iC_i$ and ABC is one of these triangles.

The proof of the second part can be stated briefly by using the formulas of the solution of 3839 [1939, 604] where the sides of the quadrilateral $[ABC, \Delta]$ are denoted by $\Delta_1, \Delta_2, \Delta_3, \Delta_4$. If ϕ_i is the orthopole of Δ_i with respect to triangle $T_i = [\Delta_j, \Delta_k, \Delta_l]$, then we find for the coordinates of ϕ_i

$$\phi_i; \quad -a, \; a(\sigma_1^{(i)} + \sigma_2^{(i)} m_i + \sigma_3^{(i)} m_i^2 + m_i^3)/(1 + m_i^2).$$

Hence the four orthopoles ϕ_i lie on the directrix of the parabola tangent to the four lines, and as is known the four orthocenters H_i of T_i lie on the same line.

Proofs of the theorem of the proposer's Note are given in the solutions of 3991 [1942, 550].

Pedals of Regular Polygons

3970 [1940, 574]. *Proposed by V. Thébault, San Sebastián, Spain*

Let $(H)\equiv A_1A_2A_3A_4A_5A_6$ and $(D)\equiv A_1\alpha_1A_2\alpha_2\cdots A_6\alpha_6$ be a regular hexagon and a regular dodecagon inscribed in a circle (O). Show that: (1) The Simson lines Δ_1, Δ_2 of any point M of (O) with respect to the triangles $A_1A_3A_5$, $A_2A_4A_6$ are perpendicular and intersect at the midpoint of MO. (2) The consecutive sides of the pedal (H') of M with respect to (H) are parallel to Δ_1, Δ_2. (3) The opposite sides of the pedal (D') of M with respect to (D) are parallel to the bisectors of the angles between Δ_1 and Δ_2. (4) Two sides of (D'), separated by a side, are perpendicular. (5) If we denote by S_6, S_{12}, Σ_{12} the areas of (H), (D), (D') then $\Sigma_{12}=S_6+S_{12}/2$. (6) Extend (3) and (4) to pedal polygons of M with respect to a regular polygon of $6k$ sides, k being any integer.

Solution by J. W. Clawson, Ursinus College. We place the polygons (H) and (D) in the circle with unit radius and center at the point O in the complex plane, so that $A_j=\text{cis}\,(j-1)\pi/3$, $j=1, 2, 3, 4, 5, 6$, where $\text{cis}\,\theta=\cos\theta+i\cdot\sin\theta$. Then $\alpha_j=\text{cis}\,(2j-1)\pi/6$.

(1) Taking M to be the turn t, the Simson line of $A_1A_3A_5$ is $t^2z-t\bar{z}=(t^3-1)/2$, where $\bar{z}$ is the reflection of z in the axis of reals, *i.e.*, if $z=r\cdot\text{cis}\,\theta$, $\bar{z}=r/\text{cis}\,\theta$.

Similarly the Simson line of $A_2A_4A_6$ is $t^2z+t\bar{z}=(t^3+1)/2$.

These lines are mutually perpendicular and intersect at $t/2$, which is the midpoint of MO.

(2) Let the foot of the perpendicular from M to A_jA_{j+1} be B_j. Then B_j is

$$\frac{1}{2}\left[\text{cis}\,(2j-1)\frac{\pi}{6}+t-\text{cis}\,(2j-1)\frac{\pi}{3}\cdot 1/t\right].$$

The equation of B_jB_{j+1} turns out to be $tz-\text{cis}\,\pi j\cdot\bar{z}=tB_j-\text{cis}\,\pi j\cdot\overline{B}_j$. If $j=1$, 3 or 5, the left-hand side of this equation becomes $tz+\bar{z}$; if $j=2$, 4 or 6, it becomes $tz-\bar{z}$.

(3) The bisectors of the angles between Δ_1 and Δ_2 are $2t^2z+2it\bar{z}=t^3+i$ and $2t^2z-2it\bar{z}=t^3-i$.

The equations of $A_j\alpha_j$ and of α_jA_{j+1} are

$$z+\text{cis}\,(4j-3)\frac{\pi}{6}\cdot\bar{z}=\frac{\sqrt{6}+\sqrt{2}}{2}\cdot\text{cis}\,(4j-3)\frac{\pi}{12},$$

$$z+\text{cis}\,(4j-1)\frac{\pi}{6}\cdot\bar{z}=\frac{\sqrt{6}+\sqrt{2}}{2}\cdot\text{cis}\,(4j-1)\frac{\pi}{12}.$$

Call the foot of the perpendiculars from M to $A_j\alpha_j$ and to α_jA_{j+1}, C_j and C_j'. Then C_j is

$$\frac{1}{2}\left[\frac{\sqrt{6}+\sqrt{2}}{2}\,\text{cis}\,(4j-3)\frac{\pi}{12}+t-\text{cis}\,(4j-3)\frac{\pi}{6}\cdot 1/t\right]$$

and the equation of C_jC_j' turns out to be

$$tz - \operatorname{cis}(2j-1)\frac{\pi}{2}\cdot\bar{z} = tC_j - \operatorname{cis}(2j-1)\frac{\pi}{2}\cdot\bar{C}_j.$$

If $j=1$, 3 or 5 the left hand side of this equation is $tz-i\bar{z}$; if $j=2$, 4 or 6 it is $tz+i\bar{z}$.

The equation of $C_j'C_{j+1}$ is $tz-\operatorname{cis} j\pi\cdot\bar{z}=tC_j'-\operatorname{cis} j\pi\cdot\bar{C}_j'$. If $j=1$, 3 or 5, the left hand side of this equation is $tz+\bar{z}$; if $j=2$, 4 or 6, it is $tz-\bar{z}$. Thus three sets of opposite sides of the pedal (D') appear to be parallel to the bisectors of the angles between Δ_1 and Δ_2; while the other three sets of opposite sides appear to be parallel to the lines Δ_1 and Δ_2 themselves.

(4) This follows immediately.

(5) S_6 is $3\sqrt{3}/2$ and S_{12} is 3. To find $\sum_{12}$ we set up the value of

$$\tfrac{1}{2}i[C_1C_2' - C_1'C_2 + \cdots + C_6C_1' - C_6'C_1].$$

This is laborious but finally yields $6(\sqrt{3}+1)/4$.

(6) Let us call the vertices of the polygon of $6k$ sides, beginning at A_j.

$$A_j, \beta_j, \beta_j', \beta_j'', \cdots, \beta_j^{(k-2)}, A_{j+1}, \cdots.$$

Then β_j is cis $(jk-k+1)\pi/3k$, β_j' is cis $(jk-k+2)\pi/3k$, and so on. Working as in (3), we find that the equations of the sides of the pedal polygon D_jD_j' and $D_j'D_j''$ are

$$tz - \bar{z}\cdot\operatorname{cis}(jk-k+1)\pi/k = tD_j - \bar{D}_j\cdot\operatorname{cis}(jk-k+1)\pi/k,$$
$$tz - \bar{z}\cdot\operatorname{cis}(jk-k+2)\pi/k = tD_j' - \bar{D}_j'\cdot\operatorname{cis}(jk-k+2)\pi/k.$$

Hence, opposite sides of the polygon are mutually perpendicular; sides separated by k sides are mutually perpendicular; sides ending at the original vertices $A_1, A_2, \cdots$ are parallel to the lines Δ_1 and Δ_2, while the other sides are parallel to the $2k$ lines which divide the angles between Δ_1 and Δ_2 into k equal parts.

Editorial Note. Let P_n be a regular polygon inscribed in circle (O) with the vertices $1, 2, 3, \cdots, n$, and let M be a point on (O) say on the arc $\widehat{12}$, not at an end point. The projection of M on the chord ij is denoted by (i,j); then the pedal P_n' of M with respect to P_n has the vertices $(1, 2), (2, 3), \cdots, (i, i+1), (i+1, i+2), (i+2, i+3), \cdots, (n, 1)$. The area of triangle $(i, i+1)M(i+1, i+2)$ is

$$(R^2 \sin 2\phi)/2[\cos^2\phi + \tfrac{1}{2}\cos 2\phi - 2\cos^2\phi\cos(2\theta_i + 2\phi) + \tfrac{1}{2}\cos(4\theta_i + 4\phi)],$$

where R is the radius of (O), 2ϕ is the angle subtended at O by each side of P_n, $2\theta_i$ is the angle subtended by the chord Mi at O. The area of P_n' is the sum of the areas of the n triangles with $i=2, 3, 4, \cdots, n, 1$, where $\theta_i=(i-2)\phi+\alpha$, angle $MO2=2\alpha$, $n\phi=\pi$. The sum of each of the two last terms is zero, so that denoting by P_n' its area, we have

$$P_n' = (\tfrac{3}{2} - 2\sin^2\phi)P_n \tag{1}$$

where n may be even or odd. If $n=2m$, where m may be even or odd,

$$P'_{2m} = \tfrac{1}{2}P_{2m} + \cos 2\phi P_{2m} = \tfrac{1}{2}P_{2m} + P_m. \tag{2}$$

We now consider the relation between the polygons P_{2m}' and P_m'. The points $(i, i+1)$, $(i+1, i+2)$, $(i+2, i)$ are collinear, being the projections of M on the sides of triangle $(i, i+1, i+2)$; the chord $M\,i+2$ of (O) subtends right angles at $(i, i+2)$, $(i+1, i+2)$, $(i+2, i+3)$, $(i+2, i+4)$, where the points $(i+2, i+3)$, $(i+3, i+4)$, $(i+4, i+2)$ are collinear for the same reason as above. From this it results that the external angles of P_{2m}' at its vertices $(i+1, i+2)$ and $(i+2, i+3)$ are each 3ϕ, also side $(i, i+2)$, $(i+2, i+4)$ of P_m' is parallel to side $(i+1, i+2)$, $(i+2, i+3)$ of P_{2m}', forming with it an isosceles trapezoid. In a similar manner we find that the external angles of P_m' at its vertices $(i, i+2)$ and $(i+2, i+4)$ are each 6ϕ. There is a second polygon P_m'' inscribed in P_{2m}' with the two consecutive sides $(i-1, i+1)$, $(i+1, i+3)$ and $(i+1, i+3)$, $(i+3, i+5)$ which are parallel respectively to the sides of P_{2m}' $(i, i+1)$, $(i+1, i+2)$ and $(i+2, i+3)$, $(i+3, i+4)$. Thus the angle between corresponding sides of P_m' and P_m'', say $(i, i+2)$, $(i+2, i+4)$ and $(i+1, i+3)$, $(i+3, i+5)$ is 3ϕ in the positive sense from the first to the second.

We consider now the case $m=6$. For the equilateral triangle with the vertices 1, 5, 9 inscribed in (O) the point M is the focus of a parabola tangent to its sides. Here $3\phi=\pi$, and the pedal P_3' of M with respect to this triangle is the straight line of (1, 5), (5, 9), (9, 1) which is the vertex tangent of the parabola. Since its directrix passes through the orthocenter O of the triangle, the radius OM is bisected by P_3'. Similarly, the pedal P_3'' of M with respect to triangle 3, 7, 11, that is the straight line of (3, 7), (7, 11), (11, 3) bisects OM. The two triangles determine a hexagon P_6 with the pedal P_6' whose exterior angles are $\pi/2$. Hence the angle between P_3' and P_3'' is $\pi/2$. One vertex of P_n', $n>3$, requires special consideration; it is vertex (1, 2) if M lies on arc $\overset{\frown}{1\ 2}$ of (O) but not at a vertex. This vertex lies inside (O) while $(n, 1)$ and (2, 3) lie outside. The chord M 2 of (O) subtends right angles at (1, 2), $(2, n)$, (2, 3) where (2, 3), (3, 1), (1, 2) are collinear. The angle $(2, n)$ 2 (2, 3) $=3\phi$, and hence angle $(n, 1)$, (1, 2), (2, 3) $=3\phi$. Thus M lies inside the triangle with vertices $(n, 1)$, (1, 2), (2, 3) and angle $(n, 1)$, (1, 2) $M=2\phi-\alpha$.

The regular polygon P_{6k} leads to the pedal P_{6k}' which has $\pi/2k$ for the external angle at its vertices, and we consider the $k+1$ consecutive sides $(2k, 2k+1)$, $(2k+1, 2k+2)$, $(2k+1, 2k+2)$, $(2k+2, 2k+3)$, $\cdots$, $(3k, 3k+1)$, $(3k+1, 3k+2)$. The triangle with vertices $1, 2k+1, 4k+1$ is equilateral and its pedal P_3' is the line of collinearity of $(1, 2k+1)$, $(2k+1, 4k+1)$, $(4k+1, 1)$ which we call Δ_1. Similarly, the equilateral triangle $k+1, 3k+1, 5k+1$ gives the pedal P_3'', or Δ_2. From the above Δ_1 and Δ_2 are perpendicular and intersect at the midpoint of OM. The chord M $2k+1$ of (O) subtends right angles at $(2k, 2k+1)$, $(2k+1, 2k+2)$, $(2k+1, 4k+1)$, and we have angle $(1, 2k+1)$, $(2k+1, 4k+1)$, $(2k, 2k+1)$ = angle $(1, 2k+1)$, $(2k+1)$, $(2k, 2k+1) = (2k-1)\phi$,

where $\phi=\pi/6k$. Similarly, angle $(2k+1,\ 2k+2)$, $(1,\ 2k+1)$, $(2k+1,\ 4k+1)$ $=$angle $(2k+2)$, $(2k+1)$, $(2k+1,\ 4k+1)=(2k-1)\phi$; and hence $(1,\ 2k+1)$, $(2k,\ 2k+1)$, $(2k+1,\ 2k+2)$, $(2k+1,\ 4k+1)$ is an isosceles trapezoid, and the first of the above $k+1$ consecutive sides of P_{6k}' is parallel to Δ_1; the next side is parallel to Δ_1 turned through the angle $\pi/2k$; the third is parallel to Δ_1 turned through the angle π/k; and so on, the last is parallel to Δ_1 turned through the angle $\pi/2$, or it is parallel to Δ_2. This may be continued and we come to $(4k,\ k+1)$, $(4k+1,\ 4k+2)$ which is parallel to Δ_2 turned through the angle $\pi/2$, or is parallel to Δ_1, and this side is opposite to $(2k,\ 2k+1)$, $(2k+1,\ 2k+2)$. Here we have k pairs of perpendiculars similar to Δ_1, Δ_2 each pair intersecting at the midpoint of OM.

For related theorems see the solutions of 3861 [1940, 118] and of 3886 [1940, 405]. For completeness we give another area formula which is similarly obtained. Here M is any point at the constant distance r from O and we denote by P_n and $\overline{P}_n$ the areas of the regular polygons of n sides inscribed respectively in the circles of radii R and r, and by P_n' the area of the pedal of M with respect to P_n.

$$P_n' = \tfrac{1}{2}P_n + \tfrac{1}{2}(P_n + \overline{P}_n)\cos 2\phi.$$

Special Circulants

3994 [1942, 341] (corrected). *Proposed by C. E. Springer, University of Oklahoma*

If

$$a_{11} = a_{22} = a_{33} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 0 \pmod 3;$$

$$a_{12} = a_{23} = a_{31} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 1 \pmod 3;$$

$$a_{13} = a_{32} = a_{21} = \sum_j \binom{K}{j}(n-1)^{K-j}, \qquad j \equiv 2 \pmod 3;$$

show that the determinant

$$|a_{ij}| = [(n-1)^3 + 1]^K.$$

Solution by John Williamson, Queens College. The determinant $|a_{ij}|$ is a circulant and as such has the value

$$\prod_{i=1}^{3} (a_{11} + \omega_i a_{12} + \omega_i^2 a_{13}) \tag{1}$$

where $\omega_1, \omega_2, \omega_3$ are the cube roots of unity. Moreover

$$(x^3+1)^K = \prod_{i=1}^{3}(x+\omega_i)^K$$

$$= \prod_{i=1}^{3}\left\{\sum_j \omega_i^j \binom{K}{j} x^{K-j}\right\}$$

$$= \prod_{i=1}^{3}(a_{11}+\omega_i a_{12}+\omega_i^2 a_{13}), \quad \text{if} \quad x = n-1,$$

$$= |a_{ij}| \quad \text{by} \quad (1).$$

The proof of the following generalization is exactly similar to the above. *Let* $|a_{rs}|$ *be a circulant of odd order* n, *where*

$$a_{rs} = \sum_j \binom{K}{j} x^{K-j}, \qquad j \equiv s-r \bmod n. \tag{2}$$

Then

$$|a_{rs}| = (x^n+1)^K.$$

The corresponding theorem for n even or odd is the following. *Let* $|b_{rs}|$ *be a circulant of order* n, *where*

$$b_{rs} = \sum_j (-1)^j \binom{K}{j} y^{K-j}, \qquad j \equiv s-r \bmod n. \tag{3}$$

Then

$$|b_{rs}| = (y^n-1)^K.$$

If $y=-x$ and n is odd it follows from (2) and (3) that $b_{rs}=(-1)^K a_{rs}$ and therefore that $|b_{rs}| = (-1)^K |a_{rs}|$. Since $(y^n-1)^K = (-1)^K(x^n+1)^K$ the theorem for n odd may be deduced from this last one.

Solved also in a different manner by the proposer with the remark that n and K of the problem are integers greater than unity and that a generalization of the result is possible in which r replaces 1 and the odd integer p replaces 3.

Editorial Note. The generalization mentioned by the proposer may be obtained by setting

$$a_u = \sum (-1)^j c^j \binom{K}{j} y^{K-j}, \qquad j \equiv u-1 \bmod n,$$

where K is a positive integer and c is not zero. Then

$$\sum_{u=1}^{n} \omega_i^{u-1} a_u = (y-\omega_i c)^K,$$

where $\omega_1, \omega_2, \cdots, \omega_n$ are the distinct nth roots of unity. The circulant C_n whose first row is $a_1, a_2, \cdots, a_n$ is such that, if we replace its first column

c_1 by $c_1+\omega_i c_2+\omega_i^2 c_3+\cdots+\omega_i^{n-1}c_n$, it will be seen that it is divisible by $(y-\omega_i c)^K$, and hence by the product of the n such factors $(y^n-c^n)^K$. Since the term of C_n of the highest degree in y is y^{nK}, we have

$$C_n = (y^n - c^n)^K.$$

There are several proofs of the factorization theorem for the general circulant C_n. One is similar to the above where the elements a_u are regarded as n independent variables instead of polynomials in the single variable y, see Scott's *The Theory of Determinants*, 1904, pp. 102, 103; on page 104 is an example somewhat similar to the one of this problem.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

Professor W. D. Cairns has been appointed visiting professor of mathematics at the University of New Mexico.

Dr. Nancy Cole of Sweet Briar College is now a visiting assistant professor at Kenyon College.

W. B. Coleman of the Agricultural and Mechanical College of Texas is now a second lieutenant in the Army Air Corps stationed at the San Antonio Aviation Cadet Center.

Assistant Professor J. H. Curtiss and Dr. Joseph Lehner have left Cornell University. The former has been commissioned lieutenant (j.g.) in the Naval Reserve and the latter is in the employ of the Kellex Corporation of New York City.

Dr. H. H. Goldstine of the University of Michigan is now a first lieutenant in the Army Air Force stationed in the ordnance department at Philadelphia.

Dr. Mary C. Graustein of Radcliffe College has been appointed to an assistant professorship at Oberlin College.

Dr. D. W. Hall of Brown University has been appointed to an assistant professorship at the University of Maryland.

Assistant Professor R. N. Johanson of Bradley Polytechnic Institute is now an assistant professor at Hamilton College.

Associate Professor Fritz John of the University of Kentucky is now in the Ballistic Research Laboratory at Aberdeen Proving Ground.

Assistant Professor E. M. Justin has a leave of absence from Case School of Applied Science in order to serve as an instructor of airplane drawing and mathematics at the University of California.

E. C. Molina of the Bell Telephone Laboratories has retired.

Dr. J. M. H. Olmsted of the University of Minnesota has been promoted to an assistant professorship.

Dr. Grace S. Quinn has been appointed associate in mathematics at George Washington University.

Professor L. W. Sheridan is on leave of absence from the College of Mount St. Vincent and is doing research in meteorology at the central office of the United States Weather Bureau in Washington, D. C.

Professor A. J. Smith of Susquehanna College has been appointed to an assistant professorship at the College of William and Mary.

Assistant Professor F. H. Steen of Georgia School of Technology has been appointed to an assistant professorship at Allegheny College.

The following appointments to instructorships have been announced:

Army Air Forces Technical Training School at Scott Field, Illinois: K. F. McLaughlin (assistant instructor)
Brown University: D. A. Jonah, M. E. Munroe, P. C. Rosenbloom
Cornell University: F. W. Dittman (part-time)
Grays Harbor Junior College (Aberdeen, Washington): Miss Ruth E. Porter
Hamilton College: Dr. V. O. McBrien
University of Minnesota: W. D. Munro
Queens College: Dr. Deborah M. Maria

Professor W. M. Carruth of Hamilton College died January 23, 1943. He was a charter member of the Association.

Dr. W. A. Granville, vice-president of the Washington National Insurance Company and author of one of the most widely used calculus texts of our generation, died February 4, 1943 at the age of seventy-nine. He was a charter member of the Association.

Professor Emeritus L. M. Passano of Massachusetts Institute of Technology died January 30, 1943.

Assistant Professor N. S. Risley of Fenn College died December 30, 1942.

A copy of G. N. Watson's *Theory of Bessel's Functions* is urgently needed in connection with important war research. Anyone who has a copy he would be willing to sell, or to lend for the duration, is requested to communicate with Dr. Warren Weaver, Chief of the Applied Mathematics Panel, National Defense Research Committee, Room 5500, 49 West 49th Street, New York, N. Y.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, University of New Mexico, Albuquerque, New Mexico.

FROM SELECTIVE SERVICE

Occupational Bulletin Number 11 was amended March 1, 1943. The recommendation of the new bulletin in regard to the deferment of graduate students is essentially the same as that of December 14, 1942 except for a significant change in wording; the latest bulletin states that graduate students in some twenty technical fields, including mathematics, "*should* be considered for occupational classification" whereas the previous wording has been "*may* be considered for occupational classification."

The new bulletin announces a liberalization of draft deferment policies in regard to undergraduate students. It specifies that "a student in undergraduate work in any of the scientific and specialized fields listed *should* be considered for occupational classification if he is a full-time student in good standing in a recognized college or university and if it is certified by the institution as follows: (1) That he is competent and gives promise of successful completion of such course of study, and (2) That if he continues his progress he will graduate from such course of study on or before July 1, 1945."

RETRAINING PROGRAMS FOR TEACHERS

The magnitude of the Army-Navy Educational Program emphasizes the shortage of mathematics and physics teachers available for instructional purposes upon the college level. An acute situation has existed in physics for many months, and now it is generally acknowledged that the supply of mathematicians is quite inadequate. Consequently, many departments of mathematics are following the advice issued to departments of physics by Dr. Homer L. Dodge, Director of the Office of Scientific Personnel, National Research Council. He states that "many departments of physics have available some graduate and senior students who can immediately be called upon to serve as teachers. Most departments, however, will have to meet most or all of the demands for staff expansion by recruitment from the staffs of other departments and outside sources. The sooner this is realized the better, for it is essential that such persons enjoy a period of thorough training."

One of the first programs for the retraining of teachers was inaugurated at New York University under Mr. M. C. Giannini, Director of the ESMWT Program in the College of Engineering, University Heights, New York. The courses have been open to all faculty members in New York University and to faculty members of neighboring institutions, including high schools. The first retraining course in mathematics comprised the subject matter of college algebra

and trigonometry. This was a sixty-hour program, meeting in the late afternoon for three, two-hour sessions every week. Twenty-seven trainees enrolled in the course, and twenty-two successfully completed the program. The trainees included several persons with the Ph.D. degree and two chairmen of departments; they came from such departments as modern language, psychology, and philosophy. At the completion of the first course, twenty members of the class petitioned for a continuation of the training, and a sixty-hour course in analytical geometry has been started. It is expected that many of these trainees, all of whom did exceptional work, will be used in the mathematics department.

THE CURTISS-WRIGHT ENGINEERING CADETTE PROGRAM

The Curtiss-Wright Corporation recently introduced a ten-months engineering training course for women. Women to be trained have been assigned to the following institutions: Cornell University, Iowa State College, University of Minnesota, Pennsylvania State College, Purdue University, and the University of Texas. In addition to mathematics, the courses to be studied are job terminology and specifications, aircraft drawing and design, elementary mechanics, and aircraft materials.

The outline of the course in engineering mathematics follows:

1. *Logarithms* (12 hours): Definition of logarithms; use of common logarithms in multiplication, division, powers, and roots; natural logarithms and conversions.

2. *Slide Rule* (12 hours): Addition and subtraction with simple scales; relationship of the slide rule to logarithms; multiplication, division, powers, and roots by slide rule; rules for decimal points.

3. *Trigonometry* (60 hours): (a) Definitions; functions; solution of right triangles (42 hours). (b) Solution of general triangles; functions of double angles (18 hours).

4. *Areas and Volumes* (18 hours): Areas of plane figures; volumes and areas of surfaces of prisms, cones, pyramids, frustums; volumes of solids of revolution (including Simpson's rule and the rule of Pappus); centroids of simple areas and volumes; use of planimeter.

5. *Review of Algebra* (30 hours): Topics through quadratic equations.

6. *Analytical Geometry* (33 hours): Rectangular coordinate systems; properties of conic sections; brief treatment of polar coordinates.

7. *Aircraft Problems* (33 hours): Graph paper, rectangular and polar coordinates; logarithmic papers; construction of charts; graphical solution of equations; curve plotting; plotting of normals and tangents to irregular curves; construction of graphs as computing aid.

ENGINEERING, SCIENCE, AND MANAGEMENT WAR TRAINING

Many of the war training courses in mathematics have been under the sponsorship of ESMWT. This has been especially true in various industrial regions. Thus the following information in regard to ESMWT furnished by the

Federal Security Agency, U. S. Office of Education, will be of interest to mathematicians.

Origin and Purpose. The Engineering, Science, and Management War Training program is the successor of the Engineering, Science, and Management Defense Training program of 1941–42, and the Engineering Defense Training program of 1940–41. ESMDT was established to provide short courses of college grade, designed to meet the shortage of engineers, chemists, physicists, and production supervisors in activities essential to the national defense. Under the program, courses were conducted at 196 institutions at a cost of approximately $18,000,000. About 438,000 trainees were enrolled. The EDT program of 1940–41 was authorized to provide short courses of college grade in engineering only. These were given in 144 engineering institutions with enrollments of about 120,000. The cost of EDT was approximately $6,140,000.

Present Status. On June 30, 1942, in Public Law 647—77th Congress, Second Session, a sum of $30,000,000 was appropriated to continue the training of engineers, chemists, physicists, and production supervisors for war service throughout the fiscal year ending June 30, 1943.

Organization of Courses. Tax-exempt colleges and universities offering recognized degrees with majors in engineering, chemistry, physics, or production supervision, are eligible to participate in Engineering, Science, and Management War Training, usually called ESMWT. Before a course is organized, the sponsoring college or university determines the need for the contemplated training through consultation with nearby industries and the ESMWT regional adviser for the area concerned, and prepares an estimate of the probable number and qualifications of those available for the training. If conditions are favorable, one or more short courses are designed to prepare available trainees for the jobs in which a shortage was found. Pertinent information, including estimates of costs, is sent to the U. S. Office of Education in a formal proposal to give each course that is planned. Those proposals that meet all legal, educational, and practical standards are approved, and instruction may begin as soon thereafter as qualified trainees can be enrolled. Additional proposals may be submitted as other training needs are established.

Types of Instruction. Regional differences in facilities and in war training needs dictate wide variations in the courses offered. Some are designed to prepare persons for new fields of work; others to fit those already employed in war activities for more difficult and responsible assignments. Classes may meet on the college campus or elsewhere; many institutions are conducting courses simultaneously in several cities. All classes receive personal instruction from qualified teachers, except for a few specialized correspondence courses in the subject matter of mathematics and physics which are offered to prepare high-school teachers to conduct courses in those subjects. Some courses require the full time and attention of those enrolled; others are given after working hours for the benefit of employed persons. The time required to complete a course varies from a few weeks to several months, depending upon the extent and

nature of the training. Subjects range from basic courses, such as engineering drawing and precision inspection, to refined specialties, such as geometrical optics and the X-ray diffraction analysis of metals. In general, the war training needs of an area determine the courses offered there, but some courses are conducted to meet Nation-wide needs of the armed forces and government war production activities.

Although all instruction under ESMWT is of college grade, it is not a substitute for regular courses of study leading to degrees. Regularly enrolled college students may not be admitted to any ESMWT course unless they intend, upon its completion, to enter war employment or the armed forces and do not intend to reenroll in college within the next academic year.

Costs. The Federal Government pays the cost of instruction. Therefore, no tuition or fees are required of ESMWT enrollees. Trainees must pay for textbooks and minor supplies, however, and they must defray their own living and transportation expenses.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-sixth Summer Meeting, New Brunswick, N. J., September 11–13, 1943.

Twenty-seventh Annual Meeting, Cleveland, Ohio.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN, Pittsburgh, April 17, 1943
ILLINOIS, Notre Dame, Ind., April 9–10, 1943
INDIANA, Notre Dame, April 9–10, 1943
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Ruston, La., 1943
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, BROOKLYN, N. Y., May 8, 1943
MICHIGAN, Notre Dame, Ind., April 9–10, 1943
MINNESOTA
MISSOURI, Kansas City
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, Jan. 29, 1944
OHIO, Columbus, April 1, 1943
OKLAHOMA
PHILADELPHIA, Philadelphia, Nov. 27, 1943
ROCKY MOUNTAIN, Denver, April 16–17, 1943
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 13, 1943
SOUTHWESTERN
TEXAS, Lubbock, April, 1943
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May 7, 1943

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CONTENTS

What are Eigen-Werte?	R. E. Langer	279
A Magic Cube with $6n^3$ Cells	R. V. Heath	288
Legendre Functions of the Second Kind and Related Functions	Dunham Jackson	291
Mathematics Instruction in Engineering	J. W. Cell	303
Applied Mathematics	H. B. Phillips	307
Some Historical Notes on the Cycloid	E. A. Whitman	309
Clubs and Allied Activities		316
Discussions and Notes	Richard Bellman, S. F. Bibb	318
Recent Publications		322
Problems and Solutions		326
Section Meeting		336
News and Notices		340
War Information		344
The Minnesota Program in Pre-Flight Mathematics		344
A Bibliography of Cryptography		345
Meetings of the Association and its Sections		346

MAY 1943